Part I

Introduction to Derivatives Markets

Michael Boyuan Zhu

UNIVERSITY OF WATERLOO

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The Basics

The Building Blocks

- In this course, we want to approach financial markets from a mathematical perspective.
- As with any field of mathematics, we start with some definitions of objects that describe phenomena we want to understand.
- The most basic object is an asset.
  - In this course, an asset refers to any financial instrument (e.g. bonds, stocks, options, etc.)
  - We examine assets in the context of a financial market. Within a financial market, an asset can be exchanged (i.e. bought or sold) at a certain price.
- In financial markets, the main phenomena to understand are time and randomness.

The Building Blocks - Financial Markets

We assume the following on financial markets in this course.

1. No transaction fees.
2. No bid-ask spreads.
3. One can buy any amount/share of any security.
4. One can trade instantly at any time.
5. Buying or selling a security does not change its price (no market impact).
6. No default/credit risk.
7. Allow naked short selling.
8. No information difference between investors.

$\Longrightarrow$ This is sometimes referred to as a perfect capital market.

$\Longrightarrow$ Not very realistic.

The Building Blocks - Time

- Time affects the value of assets. We denote time by the letter t or T.
  - Recall the time value of money from ACTSC 231. Money now is worth more than money later.
- The basic object that describes the effect of time is the risk-free rate. We denote the risk-free rate by the letter r.[1]
- The most basic asset that is impacted by time is a \$1 loan that repays at time T.
  - The value of the \$1 loan at time T is $\$1e^{rT}$.
  - This is also referred to as a risk-free bond. The price of this bond at time t is denoted by B_t.

[1] Unless otherwise specified, risk-free rates are continuously compounded. Both time and risk-free rate will use the same units (e.g. years, rate per annum).

The Building Blocks - Randomness

- We do not always know the value of assets in the future.
- The most basic asset that is impacted by randomness is a stock that does not pay dividends. The price of the stock at time t is denoted by S_t.
  - Future prices of a stock are uncertain. In Part I, we are in a model-free setting, which means that we do not assume anything on the distribution of S_t.
  - For $t > 0$, the price S_t is a random variable. $\{S_t\}_{t \geq 0}$ is a stochastic process, also referred to as a price process or value process.
  - S_0 is the current price, which is directly observable (and therefore not random).

The Building Blocks - Portfolios

- Investors typically combine different assets to form portfolios.
- For our purposes, a portfolio is a collection of assets in specified quantities. The price process of a portfolio $\{V_t\}_{t\geq 0}$ is the sum of the price processes of the assets in the portfolio.
- For example, suppose an investor holds a portfolio consisting of three shares of stock and a $200 bond. Then the value process of the portfolio is

$$V_t = 3S_t + 200B_t\,.$$

- If we have a positive position, we are long the asset. Recall that we assume that we allow short selling in our market. That is, we allow for negative quantities of assets. If we have a negative position in an asset, we are short the asset.
- For example, suppose an investor buys three shares of stock and sells short a $200 bond. Then the value process of the portfolio is

$$V_t = 3S_t - 200B_t\,.$$

The Building Blocks - No Arbitrage

- We now introduce the closest thing to an axiom in financial mathematics: the Principle of No Arbitrage.

"There ain't no such thing as a free lunch!"

– Milton Friedman

Definition (Arbitrage Opportunity)

An arbitrage opportunity is a portfolio price process $\{V_t\}_{t \geq 0}$ such that:

(i) $V_0 \leq 0$, and

(ii) $V_T > 0$, for some time $T > 0$.

Definition (Principle of No Arbitrage)

There are no arbitrage opportunities in this market.

A First Result - The Law of One Price

- There can be some debate on whether or not the Principle of No Arbitrage is realistic. However, from a mathematical perspective, this principle is fundamental, since it allows us to prove some important results.
- An immediate consequence of this is the Law of One Price.

Proposition (Law of One Price)

Suppose the market is arbitrage-free (i.e. the Principle of No Arbitrage holds). Then if two portfolios have exactly the same payoff, then they must have the same price.

A First Result - The Law of One Price - Proof

Proposition (Law of One Price (restated))

Suppose the market is arbitrage-free. Let $\{V_t\}_{t\geq 0}$ *and* $\{X_t\}_{t\geq 0}$ *be two portfolios such that* $V_T = X_T$ *for some* $T > 0$*. Then* $V_0 = X_0$*.*

Proof.

We prove by contradiction. Suppose (for the sake of contradiction, WLOG) that $V_0 > X_0$. Construct a portfolio by longing X_0 and shorting V_0, and then investing the proceeds in a risk-free bond. Specifically, we form the portfolio:

$$Z_t := X_t - V_t + (V_0 - X_0)B_t\,, \text{ for } t \geq 0\,.$$

Proof (cont'd).

At time 0:

$$Z_0 = X_0 - V_0 + (V_0 - X_0)\underbrace{B_0}_{=1} = 0\,.$$

At time T:

$$Z_T = X_T - V_T + (V_0 - X_0)B_T = \underbrace{(V_0 - X_0)}_{>0}\underbrace{B_T}_{=e^{rT}} > 0\,.$$

Hence, $\{Z_t\}_{t\geq 0}$ is an arbitrage portfolio. This violates the Principle of No Arbitrage: a contradiction. □

$\Longrightarrow$ We will often see this kind of construction when trying to build an arbitrage portfolio: buy cheap, sell high, and invest the proceeds.

Some Other Results

- From the Principle of No Arbitrage, we can also prove the following. The proofs are left as exercises.

Proposition

If a portfolio has a higher payoff than another, then it must have a higher price.

Proposition

The risk-free rate is unique.

$\Rightarrow$ Interestingly, this means that if we have $r > 0$, then $r = 0$ is not an option. If you are holding cash and doing nothing with it, you are getting arbitraged!

Summary and What's Next

- So far, we have introduced the building blocks of mathematical finance and proved our first result.
- Next, we are going to introduce the concept of derivatives: more complicated assets whose payoffs depend on other underlying assets.
- Our main goal in this course is to determine the price of different derivative instruments.
- In Part I, we do this in a model-free setting. We do not assume anything on the distribution of the stock! It turns out there are quite a few things we can say about prices, even without any models of stock prices.

Your First Derivative: Forwards (and Futures)

Derivatives

Definition (Derivatives, *Wikipedia* Version)

In finance, a derivative is a contract that derives its value from the performance of an underlying entity.

- This underlying entity can be a stock, index, or interest rate, and is often called the underlying asset, or more simply, the underlying.
- Some derivatives are also referred to as contingent claims. Here, the "claim" refers to the payoff of the instrument, which is "contingent" on the underlying asset.

Derivatives

Definition (Derivatives)

A derivative is a financial contract whose value at an expiration date (maturity) T is "derived" exactly from (i.e. a function of) the market prices of more basic underlying primitive instruments or other derivatives, up to and including time T.

Primitive Instruments (underlying):

- Stocks
- Currencies
- Interest rates
- Indices and ETFs
- Commodities
- Bonds

Derivatives:

- Forwards & Futures
- Options (Calls, Puts, Caps, Floors, Bond Options, Swaptions . . .)
- Credit derivatives
- Swaps

Derivatives

Derivatives are used:

- to manage risk:
  - e.g. a pension fund invested in a broad market index can use derivatives to obtain downside protection.
  - e.g. an airline company can use derivatives to put a ceiling on the future price of jet fuel.
- as an important part of compensation:
  - e.g. executives receive stock options that become valuable if their company does well. This provides an incentive.
- for speculation:
  - e.g. the use of derivatives can magnify financial consequences. One can obtain large exposures with relatively little capital.

Derivatives

"We all know that no one understands derivatives, except perhaps Mr. Buffett and the computers who created them."

– Richard Dooling, in the New York Times

"In our view, derivatives are financial weapons of mass destruction, carrying dangers that, while now latent, are potentially lethal. We view them as time bombs, both for the parties that deal in them and the economic system."

– Warren Buffett, 2002 Berkshire Hathaway annual report

Forwards - Introduction

- A forward contract is the simplest derivative we will study.

Definition (Forward contracts)

A forward contract is an agreement to buy or sell an asset at a future time T for a predetermined price K.

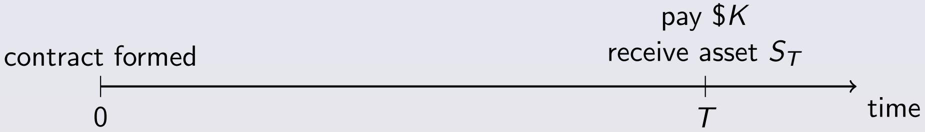
$\Rightarrow$ The delivery price is determined so that the value of the contract at initiation is zero.

- The time at which the asset is delivered (T) is called the expiration date.
- The price that will be paid (K) is called the forward price.
  - $\Rightarrow$ This is NOT the price that is paid at the initial time. There is no exchange of money at time 0!
- In essence, the forward is a binding contract that delays the exchange of an asset into the future.

Forwards - Payoff

- The party that agrees to buy the asset is said to be long the forward contract.
- The party that agrees to sell the asset is said to be short the forward contract.

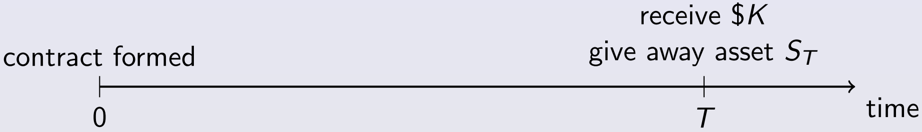
The following diagram depicts a long forward with expiration date T with an underlying stock S:



$\Rightarrow$ Since the asset is worth S_T at time T, the payoff at time T is $S_T - K$.

Forwards - Payoff

- The following diagram depicts a short forward with expiration date T with an underlying stock S:



$\Rightarrow$ The payoff at time T is $K - S_T$.

Forwards - Determining the Forward Price

- We now wish to determine the forward price. That is, what should a fair value of K be?

Remark (Forward Price vs. Price of the Forward)

Compared to options later on, we do this "backwards". Given that the price of the forward is zero, we try to find a fair K (which functions like a strike price). For options, we are given the strike K, and then we try to find a fair price of the option.

- We can do this by constructing a replicating portfolio (i.e. a portfolio with the same payoff). We then apply the Law of One Price to determine the appropriate forward price.

Forwards - Determining the Forward Price

- Consider the following portfolio:
  (i) Long one stock at time 0.
  (ii) Borrow $\$Ke^{-rT}$ at time 0 (i.e. sell a bond).
- The cash flows of this portfolio are given in the following table:

	Cash flow at $t = 0$	Cash flow at $t = T$
1 long share	$-S_0$	$+S_T$
Borrowing $\$Ke^{-rT}$	$+Ke^{-rT}$	$-K$
TOTAL	$Ke^{-rT} - S_0$	$S_T - K$

- Note that the cash flow at time T is precisely the payoff of a long forward!

Forwards - Determining the Forward Price

- Since the price of a long forward is 0, then by the Law of One Price, the price of this portfolio is also 0. So we have

$$Ke^{-rT} - S_0 = 0 \implies K = S_0 e^{rT}.$$

$\Rightarrow$ Therefore, the fair forward price for a non-dividend paying stock is $S_0 e^{rT}$.

Forwards - Determining the Forward Price - Version 2

- We can arrive at the same conclusion by directly applying the Principle of No Arbitrage. In this approach, we try to construct a zero-cost portfolio. Consider the following portfolio:

  (i) Long one stock at time 0.

  (ii) Borrow S_0 at the risk-free rate.

  (iii) Short one forward contract on the stock.

- The cash flows are:

	Cash flow at $t = 0$	Cash flow at $t = T$
Borrowing S_0	$+S_0$	$-S_0 e^{rT}$
1 long share	$-S_0$	$+S_T$
1 short forward	0	$K - S_T$
TOTAL	0	$K - S_0 e^{rT}$

Forwards - Determining the Forward Price - Version 2

- Note that the payoff of this portfolio, $K - S_0 e^{rT}$ is deterministic (i.e. not random). Then the Principle of No Arbitrage implies that $K - S_0 e^{rT} \leq 0$.

  $\Rightarrow$ We can obtain the other inequality by shorting the same portfolio. Hence, in general, if a portfolio has a deterministic payoff and costs nothing, it must also pay nothing.

- Hence, we once again conclude that the fair forward price is

$$K = S_0 e^{rT} .$$

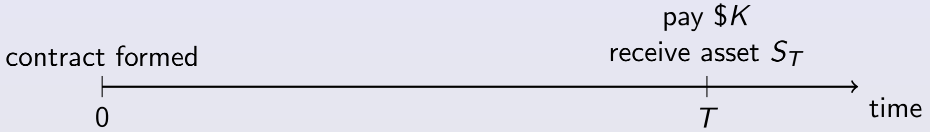
Forwards - Determining the Forward Price - Dividends

- Dividends are the payments made by a company to its shareholders.
- Dividends in practice are usually discrete, which means that they are paid at discrete time intervals (e.g. quarterly, yearly).
- The level of dividend can vary greatly. Many tech stocks do not pay a dividend (e.g. Amazon (AMZN) and Tesla (TSLA)). Financials and utilities tend to pay a higher dividend (e.g. TD Bank (TD, 4.64%) and Enbridge (ENB, 7.46%)).[2]
- In this course, we also consider continuous dividends. This functions like a continuously compounded risk-free rate.
  - We assume that any continuous dividends are reinvested back into the stock.
  - Suppose a stock pays a continuous dividend at a rate δ. Then an investment of S_0 in the stock at time 0 is worth $S_T e^{\delta T}$ at time T. One share of stock at time 0 becomes $e^{\delta T}$ shares at time T.

[2] Dividend yields as of August 31, 2023.

Forwards - Determining the Forward Price - Dividends

- We now want to find a fair forward price K when the underlying stock pays a continuous dividend at rate δ. The payoff of a long forward is the same as before:



- We construct a replicating portfolio:
  (i) Long $e^{-\delta T}$ units of stock at time 0.
  (ii) Borrow $\$Ke^{-rT}$ at time 0.

Forwards - Determining the Forward Price - Dividends

- The cash flows of this portfolio are given in the following table:

	Cash flow at $t = 0$	Cash flow at $t = T$
$e^{-\delta T}$ long shares	$-S_0 e^{-\delta T}$	$+S_T$
Borrowing $\$Ke^{-rT}$	$+Ke^{-rT}$	$-K$
TOTAL	$Ke^{-rT} - S_0 e^{-\delta T}$	$S_T - K$

- By the Law of One Price, the price (at time 0) of this portfolio must be 0. Hence, we obtain the forward price

$$K = S_0 e^{(r-\delta)T} .$$

- While continuous dividends are a strange concept, they are mathematically convenient and give nice formulas. We will use continuous dividends more often than discrete dividends in this course.

Forwards vs Futures

- In the real world, forwards are over-the-counter (OTC) instruments. This means that they are privately negotiated between parties, and therefore more flexible but not regulated.
    - This also means that forwards are more susceptible to default risk.
- A future is like a forward, in that it is also a contract for a transaction at a fixed price and at a future time (and hence, has the same payoff). However, future contracts are traded on exchanges. They are less flexible, highly regulated, and less susceptible to default risk.
    - An important feature of futures is that they are marked-to-market. That means that the exchange and brokerage will keep track of the value of a future position on a daily basis. At the end of the day, the value of the position is updated to reflect the most recent market data.

Forwards vs Futures

- Recall that forwards and futures have zero price when they are negotiated. Does this mean that you can buy (or sell) as many as you want?
  - Forwards: You can... if you can find enough counterparties to agree to the contracts.
  - Futures: You can... up to a limit depending on your brokerage firm.
- Your brokerage will require you to at least have some money in the account to cover possible losses. This is called a margin.
- If the value of your account becomes too low, your brokerage will require you to put more money into the account. This is called a margin call.
- This is why marking-to-market is important! Your brokerage is checking the value of your position to make sure you do not lose too much money.

More on Futures

- You will probably find that it is difficult to trade forwards yourself.
- Futures are accessible to retail investors on certain brokerages.
- Futures markets see lots of volume and are highly liquid. Some futures are traded much more than their underlying assets are! Some reasons for this are the following:
  - Compared to buying a stock, longing a future does not require you to pay the price of the stock.
  - Compared to shorting a stock, shorting a future does not require you to have access to a stock to lend out.
  - Commissions and execution costs are low.
  - Large companies use futures extensively for hedging, managing price risk, managing forex risk...

Options: Calls and Puts

Options - Introduction

Definition (Options)

An option is a contract that gives an investor the right, but not the obligation, to buy or sell an asset at or before a future time T for a predetermined price K.

- The action of carrying out the transaction specified in the contract is called exercising the option.
- The expiration date (T) is the latest time that the option can be exercised.
- The price at which the asset will be exchanged (K) is called the strike price.

Options - Exercise Times

- An option that can only be exercised at time T (i.e. the expiration date) is called a European option.
  - Most options on stock indices (e.g. on the S&P 500) are European options.
- An option that can be exercised at any time before expiration is called an American option.
  - Most options on stocks are American options.
- An option that can only be exercised on specific pre-specified dates before expiration is called a Bermudan option.

Options - Calls and Puts

Definition (Call Option)

A call option is a contract that gives an investor the right, but not the obligation, to buy an asset at or before a future time T for a predetermined price K.

$\Rightarrow$ The payoff of a call at time T is

$$\max\{S_T - K, 0\}.$$

We will also use the notation $(S_T - K)_+$.

Definition (Put Option)

A put option is a contract that gives an investor the right, but not the obligation, to sell an asset at or before a future time T for a predetermined price K.

$\Rightarrow$ The payoff of a put at time T is $\max\{K - S_T, 0\}$ or $(K - S_T)_+$.

Options - vs Forwards and Futures

- The main difference between an option and a forward is that the holder of the option is not obliged to perform the transaction.

	Payoff at time T
Long forward	$S_T - K$
Long call	$(S_T - K)_+$
Short forward	$K - S_T$
Long put	$(K - S_T)_+$

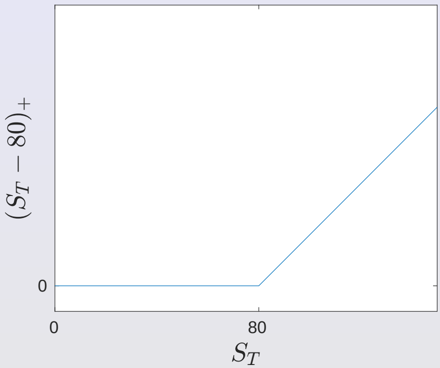
- Forwards have zero price at initiation, which determines the forward price. In contrast, the strike for an option is fixed, which determines the price of the option.

Options - Moneyness

- An option is said to be in-the-money if an immediate exercise of the option would produce a positive cash flow.
  - For example, a call option with a strike of \$80 on a stock with current price \$100 is in-the-money, since $100 - 80 = 20 > 0$.
- An option is said to be out-of-the-money if an immediate exercise of the option would produce a negative cash flow.
  - For example, a call option with a strike of \$80 on a stock with current price \$60 is out-of-the-money, since $60 - 80 = -20 < 0$.
- An option is said to be at-the-money if an immediate exercise of the option would produce no cash flow (i.e. the strike is equal to the current price of the underlying).

Options - Payoff Diagrams - Long Call

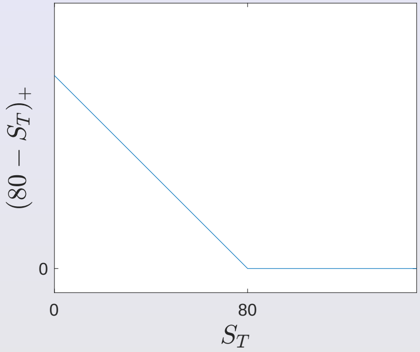
- The payoff of a long call with strike price \$80 on a stock S is given in the following graph.



⇒ The long call has unbounded potential gain. The payoff has a slope of either 0 or 1.

Options - Payoff Diagrams - Long Put

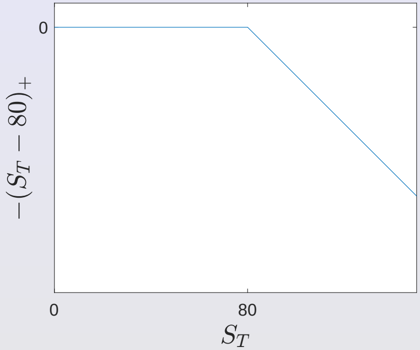
- The payoff of a long put with strike price \$80 is given in the following graph.



$\Rightarrow$ The put option has a higher payoff if the stock price decreases. Long puts are used to bet against stocks.

Options - Payoff Diagrams - Short Call

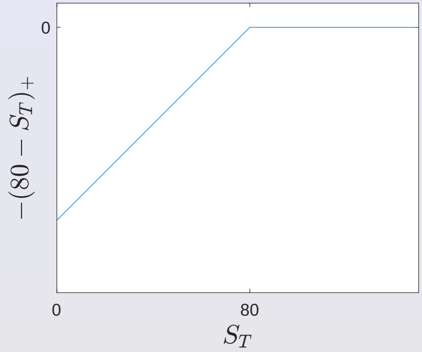
- The payoff of a short call with strike price \$80 is given in the following graph.



$\Rightarrow$ The short call has unbounded potential loss.

Options - Payoff Diagrams - Short Put

- The payoff of a short put with strike price \$80 is given in the following graph.



$\Rightarrow$ Note that short options have negative payoffs. They also have negative price, since you receive the option premium when you sell the option.

Options - Option Chains

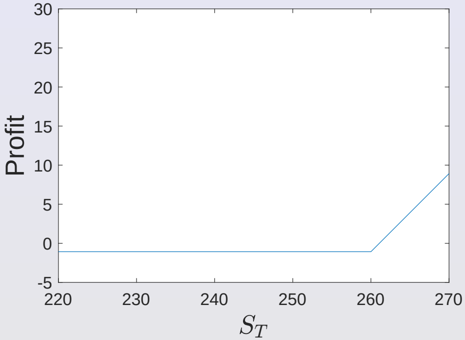
- An option chain is a listing of all available options contracts on a given security. Here's one for Tesla (TSLA), with an expiration date one week in the future:[3]

	Calls							Puts					
Exp. Date	Last	Change	Bid	Ask	Volume	Open Int.	Strike	Last	Change	Bid	Ask	Volume	Open Int.
September 8, 2023													
Sep 8	23.15	-12.90▼	22.20	23.85	186	541	222.50	0.36	+0.19▲	0.35	0.37	3772	1274
Sep 8	20.60	-13.02▼	19.85	21.50	780	1812	225.00	0.51	+0.29▲	0.47	0.50	12155	4714
Sep 8	18.35	-14.35▼	17.90	19.15	520	1018	227.50	0.68	+0.41▲	0.65	0.70	5606	1435
Sep 8	16.06	-12.60▼	15.20	16.95	1662	2262	230.00	0.94	+0.60▲	0.94	0.95	29624	8366
Sep 8	13.95	-13.85▼	13.70	14.20	1163	1370	232.50	1.28	+0.84▲	1.25	1.31	7800	3282
Sep 8	11.90	-11.98▼	11.85	12.05	1485	2879	235.00	1.73	+1.15▲	1.70	1.78	17618	5288
Sep 8	10.00	-12.09▼	9.85	10.15	2314	1672	237.50	2.36	+1.60▲	2.29	2.35	12065	4251
Sep 8	8.25	-11.22▼	8.20	8.35	7446	4715	240.00	3.09	+2.09▲	3.00	3.10	44366	9392
Sep 8	6.65	-10.50▼	6.65	6.80	6510	2112	242.50	4.00	+2.71▲	3.95	4.05	18456	2887
Sep 8	5.35	-9.87▼	5.30	5.35	24690	24213	245.00	5.15	+3.45▲	5.00	5.20	37813	5805
Sep 8	4.25	-8.75▼	4.15	4.25	23143	3134	247.50	6.45	+4.23▲	6.30	6.60	17050	2720
Sep 8	3.23	-8.10▼	3.20	3.25	57080	5805	250.00	8.04	+5.16▲	7.90	8.10	30473	9111
Sep 8	2.47	-7.22▼	2.42	2.47	21902	3422	252.50	9.80	+6.05▲	9.60	9.95	12069	3834
Sep 8	1.87	-6.20▼	1.87	1.88	53344	5737	255.00	11.67	+7.07▲	11.50	11.75	11481	3436
Sep 8	1.45	-5.25▼	1.39	1.44	17907	3754	257.50	13.67	+7.82▲	13.50	14.00	5315	3166
Sep 8	1.08	-4.36▼	1.06	1.08	79378	14012	260.00	15.82	+8.82▲	15.10	16.70	3555	2978
Sep 8	0.80	-3.65▼	0.79	0.83	13904	6233	262.50	18.30	+9.82▲	17.00	19.35	1384	1070
Sep 8	0.62	-2.98▼	0.62	0.63	42642	14204	265.00	20.45	+10.22▲	19.95	21.30	4922	1112
Sep 8	0.48	-2.44▼	0.46	0.49	8638	5817	267.50	22.84	+10.91▲	22.00	23.65	427	446

[3] Market data as of EOD Sept. 1, 2023. TSLA closing price was $245.01, down 5.06% over the trading day.

Options - Profit Example

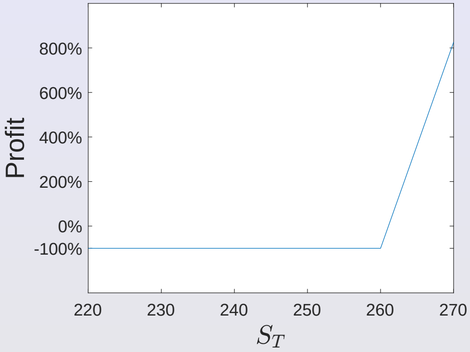
- Consider an out-of-the-money call option on TSLA with strike price \$260 and option price \$1.08. The spot price of TSLA is \$245.01. The profit diagram (i.e. payoff minus cost) is the following:



- Note that the price needs to go up to at least $260 + 1.08 = \$261.08$ to break even.

Options - Profit Example

- Profit as a percent of initial investment is given by the following:



- Be careful with options! You can easily lose all of your initial investment when dealing with out-of-the-money options.

Option Prices: Initial Bounds and Put-Call Parity

Options - Bounds on Prices

- Recall that we were able to determine a fair forward price by constructing a replicating portfolio for the forward.
- The payoff of an option cannot be replicated with just a stock and a bond. In a model-free setting, we are unable to determine a unique price of an option.
- However, we are able to obtain some results on bounds of option prices, using only the Principle of No Arbitrage.

Options - Bounds on Prices

In this section, we will use the following notation for option prices at time $t \leq T$. The strike is K and the expiration date is T.

- European options:
  - European Call: $c(K, t, T) = c_t$
  - European Put: $p(K, t, T) = p_t$
- American options:
  - American Call: $C(K, t, T) = C_t$
  - American Put: $P(K, t, T) = P_t$

$\Rightarrow$ Upper-case letters (C and P) denote prices of American options, and lower-case letters (c and p) denote prices of European options.

Options - American vs. European

Since an American option can be exercised at any time, it must always be at least as valuable as an otherwise identical European option.

Proposition (American vs. European Options)

$$C(K,t,T) \geq c(K,t,T) \quad \textit{and} \quad P(K,t,T) \geq p(K,t,T)\,.$$

Call Options - Initial Bounds

- Since the payoff of a European call option $(S_T - K)_+$ is non-negative, then we know that $c_t \geq 0$.
- The price of an American call option cannot be more than the cost of its underlying. That is, $C_t \leq S_t$.
  - Otherwise, selling the call and buying the underlying is an arbitrage opportunity.
- So far we have

$$S_t \geq C_t \geq c_t \geq 0\,,$$

but we can get a better lower bound...

European Call Option - Lower Bound

- Suppose that the underlying stock pays continuous dividends at a rate δ. Consider the following two portfolios at time t:
  (1) Long one European call option $c(K, t, T)$.
  (2) Long $e^{-\delta(T-t)}$ units of stock, and borrow $\$Ke^{-r(T-t)}$ at the risk-free rate.
- The cash flows of these strategies are:

	Cash flow at time t	Cash flow at time T	
		$S_T < K$	$S_T \geq K$
Strategy 1	$-c_t$	0	$S_T - K$
Strategy 2	$Ke^{-r(T-t)} - S_t e^{-\delta(T-t)}$	$S_T - K$	$S_T - K$

- We see that no matter what the price of the stock is at time T, Strategy 1 has a larger payoff than Strategy 2.

European Call Option - Lower Bound

- Therefore we conclude that Strategy 1 must have a higher price (or a lower cash flow). Therefore,

$$c_t \geq S_t e^{-\delta(T-t)} - Ke^{-r(T-t)}\,.$$

- Combining this with the previous inequalities, we have the following:

Proposition (Call Option Bounds)

$$S_t \geq C_t \geq c_t \geq \max\{S_t e^{-\delta(T-t)} - Ke^{-r(T-t)}, 0\}\,.$$

Remark

An easy way to remember this lower bound is by using put-call parity, which we will see later.

European Call Option - Lower Bound

- We can see these lower bounds in effect. The lower bound for in-the-money options is approximately $S_t - K$, and the bound for out-of-the-money options is 0.

	Calls							Puts					
Exp. Date	Last	Change	Bid	Ask	Volume	Open Int.	Strike	Last	Change	Bid	Ask	Volume	Open Int.
September 8, 2023													
Sep 8	23.15	-12.90▼	22.20	23.85	186	541	222.50	0.36	+0.19▲	0.35	0.37	3772	1274
Sep 8	20.60	-13.02▼	19.85	21.50	780	1812	225.00	0.51	+0.29▲	0.47	0.50	12155	4714
Sep 8	18.35	-14.35▼	17.90	19.15	520	1018	227.50	0.68	+0.41▲	0.65	0.70	5606	1435
Sep 8	16.06	-12.60▼	15.20	16.95	1662	2262	230.00	0.94	+0.60▲	0.94	0.95	29624	8366
Sep 8	13.95	-13.85▼	13.70	14.20	1163	1370	232.50	1.28	+0.84▲	1.25	1.31	7800	3282
Sep 8	11.90	-11.98▼	11.85	12.05	1485	2879	235.00	1.73	+1.15▲	1.70	1.78	17618	5288
Sep 8	10.00	-12.09▼	9.85	10.15	2314	1672	237.50	2.36	+1.60▲	2.29	2.35	12065	4251
Sep 8	8.25	-11.22▼	8.20	8.35	7446	4715	240.00	3.09	+2.09▲	3.00	3.10	44366	9392
Sep 8	6.65	-10.50▼	6.65	6.80	6510	2112	242.50	4.00	+2.71▲	3.95	4.05	18456	2887
Sep 8	5.35	-9.87▼	5.30	5.35	24690	24213	245.00	5.15	+3.45▲	5.00	5.20	37813	5805
Sep 8	4.25	-8.75▼	4.15	4.25	23143	3134	247.50	6.45	+4.23▲	6.30	6.60	17050	2720
Sep 8	3.23	-8.10▼	3.20	3.25	57080	5805	250.00	8.04	+5.16▲	7.90	8.10	30473	9111
Sep 8	2.47	-7.22▼	2.42	2.47	21902	3422	252.50	9.80	+6.05▲	9.60	9.95	12069	3834
Sep 8	1.87	-6.20▼	1.87	1.88	53344	5737	255.00	11.67	+7.07▲	11.50	11.75	11481	3436
Sep 8	1.45	-5.25▼	1.39	1.44	17907	3754	257.50	13.67	+7.82▲	13.50	14.00	5315	3166
Sep 8	1.08	-4.36▼	1.06	1.08	79378	14012	260.00	15.82	+8.82▲	15.10	16.70	3555	2978
Sep 8	0.80	-3.65▼	0.79	0.83	13904	6233	262.50	18.30	+9.82▲	17.00	19.35	1384	1070
Sep 8	0.62	-2.98▼	0.62	0.63	42642	14204	265.00	20.45	+10.22▲	19.95	21.30	4922	1112
Sep 8	0.48	-2.44▼	0.46	0.49	8638	5817	267.50	22.84	+10.91▲	22.00	23.65	427	446

American Call Option - Early Exercise

- An interesting consequence of this is that an American call option on a stock with no dividends should never be exercised early:

Proposition (Early Exercise of an American Call)

If the underlying does not pay dividends, then

$$c(K, t, T) = C(K, t, T).$$

Proof.

At time t^*, we can either exercise the American call early, or we can sell the call in the market.

- If we exercise the American call at time t^*, our payoff is $S_{t^*} - K$.

American Call Option - Early Exercise

Proof (cont'd).

- On the other hand, if we sell the call in the market, our payoff is C_{t^*}. From before, we have

$$C_{t^*} \geq c_{t^*} \geq S_{t^*} - Ke^{-r(T-t^*)} \geq S_{t^*} - K \,.$$

Hence, we are always better off selling the option instead. Therefore an American call option on a stock with no dividends has the same price as a European call option. □

- This only holds when there are no dividends! We cannot conclude that $S_{t^*}e^{-\delta(T-t^*)} - Ke^{-r(T-t^*)} \geq S_{t^*} - K$ when $\delta > 0$.
- This does not mean that holding the option and never exercising is optimal! It just means that selling is always better than exercising.

Put Options - Initial Bounds

- Since the payoff of a European put option $(K - S_T)_+$ is non-negative, then we know that $p_t \geq 0$.
- The price of an American put option cannot be more than the amount of its strike. That is, $P_t \leq K$.
  - Note that the maximum possible payout of an American put is if the stock price crashes to 0, in which case the payoff is $\$K$ upon exercising.
- So far we have

$$K \geq P_t \geq p_t \geq 0\,,$$

but again, we can get a better lower bound...

European Put Option - Lower Bound

- Suppose that the underlying stock pays continuous dividends at a rate δ. Consider the following two portfolios at time t:
  (1) Long one European put option $c(K, t, T)$.
  (2) Short $e^{-\delta(T-t)}$ units of stock, and lend $\$Ke^{-r(T-t)}$ at the risk-free rate.
- The cash flows of these strategies are:

	Cash flow at time t	Cash flow at time T	
		$S_T < K$	$S_T \geq K$
Strategy 1	$-p_t$	$K - S_T$	0
Strategy 2	$S_t e^{-\delta(T-t)} - Ke^{-r(T-t)}$	$K - S_T$	$K - S_T$

- We see that no matter what the price of the stock is at time T, Strategy 1 has a larger payoff than Strategy 2.

European Put Option - Lower Bound

- Similar to before, we conclude that Strategy 1 must have a higher price. Therefore, we have the following:

Proposition (Put Option Bounds)

$$K \geq P_t \geq p_t \geq \max\{Ke^{-r(T-t)} - S_t e^{-\delta(T-t)}, 0\}.$$

European Put Option - Lower Bound

- We can see these lower bounds in effect. The lower bound for in-the-money options is approximately $K - S_t$, and the bound for out-of-the-money options is 0.

	Calls							Puts					
Exp. Date	Last	Change	Bid	Ask	Volume	Open Int.	Strike	Last	Change	Bid	Ask	Volume	Open Int.
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American Put Option - Early Exercise

- Unlike American calls, it is always possible to optimally early exercise an American put. This is true regardless of whether or not the underlying pays dividends.
  - Technically, one can conclude that if $r = 0$ then early exercise is not optimal. However, we are not concerned with this situation, since $r = 0$ is not realistic.

Put-Call Parity

- The payoffs of calls and puts are intimately related. Consider a portfolio consisting of a long European call and a short European put with the same strike price K, the same underlying S, and the same expiration date T.

	Cash flow at time t	Cash flow at time T	
		$S_T < K$	$S_T \geq K$
Long call	$-c_t$	0	$S_T - K$
Short put	p_t	$-(K - S_T)$	0
TOTAL	$-c_t + p_t$	$S_T - K$	$S_T - K$

- In either case, the payoff for this portfolio is $S_T - K$.

$\Rightarrow$ We can create a replicating portfolio for this payoff!

Put-Call Parity

- As we have seen before, the payoff $S_T - K$ can be obtained by longing $e^{-\delta(T-t)}$ units of stock, and borrowing $\$Ke^{-r(T-t)}$.

	Cash flow at time t	Cash flow at time T
$e^{-\delta(T-t)}$ long shares	$-S_t e^{-\delta(T-t)}$	$+S_T$
Borrowing $\$Ke^{-r(T-t)}$	$+Ke^{-r(T-t)}$	$-K$
TOTAL	$Ke^{-r(T-t)} - S_t e^{-\delta(T-t)}$	$S_T - K$

- By the Law of One Price, these portfolios must have the same price. Hence, we have proved the following:

Proposition (Put-Call Parity)

For a European call and a European put on the same underlying, with the same strike K and expiration time T,

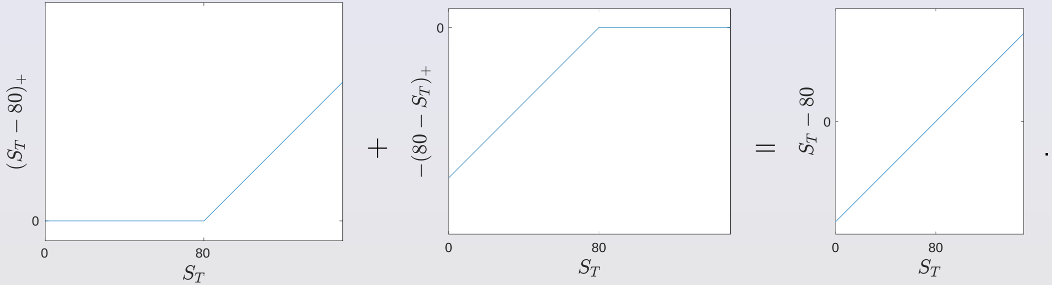
$$c_t - p_t = S_t e^{-\delta(T-t)} - Ke^{-r(T-t)}.$$

Put-Call Parity

- Mathematically, it is straightforward to verify that

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K\ .$$

- Graphically, by adding payoff diagrams, we have the following:



Put-Call Parity - American Options

- Put-call parity is a result for European options. For American options, the best we can do is the following:

Proposition (Put-Call Bounds, American Options)

For an American call and an American put on the same underlying that does not pay dividends, with the same strike K and expiration time T,

$$S_t - K \leq C_t - P_t \leq S_t - Ke^{-r(T-t)} .$$

- Note that this only works for options on a non-dividend paying stock.
- We do not get a parity result, but the difference between the left and right sides of the inequality is pretty small.

Put-Call Parity - An Example

Example

- Suppose you observe this data for TSLA and its option chain:

Stock Price	\$245
European Call price ($T = 1/52$ years, $K = \$250$)	\$3.20
European Put price ($T = 1/52$ years, $K = \$250$)	\$8.10
Risk-free Rate	2% per annum

- We can check if put-call parity holds:

$$\begin{aligned} c_0 - p_0 &\stackrel{?}{=} S_0 - Ke^{-rT} \\ 3.2 - 8.1 &\stackrel{?}{=} 245 - 250e^{-0.02/52} \\ -4.9 &> -4.904 \end{aligned}$$

$\Rightarrow$ Since put-call parity is violated, an arbitrage opportunity exists!

Put-Call Parity - An Example

Example

- To create an arbitrage portfolio, we buy the cheaper side and sell the expensive side.
- We found that

$$c_0 - p_0 > S_0 - Ke^{-rT} .$$

⇒ Buy low: Buy a share of stock, borrow money.

⇒ Sell high: Sell a call, buy a put.

Put-Call Parity - An Example

Example

- The cash flows of this strategy are:

Position	Cash flow at $t = 0$	Cash flow at $t = T$	
		$S_T < K$	$S_T \geq K$
Buy Stock	-245	S_T	S_T
Borrow $\$250e^{-0.02/52} = \$249.904$	$+249.904$	-250	-250
Sell Call	$+3.2$	0	$-(S_T - 250)$
Buy Put	-8.1	$250 - S_T$	0
TOTAL	$+\$0.004$	0	0

- We have obtained an immediate profit of \$0.004! We can scale up this position as much as we want (or as the market allows) to make free money.

Put-Call Parity - An Example

Example

- The previous example uses real stock and option prices as of September 1st 2023.
- If you can find someone to lend you money at 2%, then you can form this portfolio. Unfortunately interest rates are currently much higher.
- In practice, these opportunities are hard to find (and no-arbitrage typically holds). This is further complicated by bid-ask spreads, liquidity, market impact, etc.

Option Prices: Lipschitz Continuity, Convexity, and Other Bounds

Options - Lipschitz Continuity

- So far, we have examined relationships between different options with the same strike and maturity. However, we can also find relationships between options with different strike prices.
- For a fixed expiration date T, we will write $c(K) := c(K, t, T)$ for a European call with strike K (and similarly for the other options).

Proposition (Call Options - Decreasing 1-Lipschitz)

Both $c(K)$ and $C(K)$ are decreasing 1-Lipschitz functions of K. That is, for $0 \leq K_1 \leq K_2$,

$$0 \leq c(K_1) - c(K_2) \leq K_2 - K_1 ,$$
$$0 \leq C(K_1) - C(K_2) \leq K_2 - K_1 .$$

$\Rightarrow$ Roughly, the slopes of $c(K)$ and $C(K)$ are always between -1 and 0.

Options - Lipschitz Continuity

Proof.

The bounds for European calls follow easily from comparing payoffs. For example, $(S_T - K_1)_+ \geq (S_T - K_2)_+$, so $c(K_1) \geq c(K_2)$. The other bound is left as an exercise.

The bounds for American calls can be determined by forming a portfolio and showing that a positive payoff can always be attained. When an option is sold short, you can no longer control when the option is exercised. However, you can respond to this by exercising the other option to cover your position.

As an example, we will prove $C(K_1) - C(K_2) \geq 0$. Suppose you form a portfolio by longing $C(K_1)$ and shorting $C(K_2)$. If the counterparty never exercises $C(K_2)$, then your portfolio at time T is worth $C(K_1) \geq 0$.

Options - Lipschitz Continuity

Proof (cont'd).

Now suppose that the counterparty exercises $C(K_2)$ at some time t^*. You respond by exercising $C(K_1)$. Since the counterparty exercised, we have $S_{t^*} - K_2 \geq 0$, which implies $S_{t^*} - K_1 \geq 0$. The payoffs are:

Position	Cash flow at time t^*
Short $C(K_2)$	$K_2 - S_{t^*}$
Long $C(K_1)$	$S_{t^*} - K_1$
TOTAL	$K_2 - K_1$

Hence, this portfolio always has a non-negative payoff, so $0 \leq C(K_1) - C(K_2)$. The other bound is left as an exercise. □

Options - Lipschitz Continuity

- We obtain a similar result for put options. The proof of this result is similar.

Proposition (Put Options - Increasing 1-Lipschitz)

Both $p(K)$ and $P(K)$ are increasing 1-Lipschitz functions of K. That is, for $0 \leq K_1 \leq K_2$,

$$0 \leq p(K_2) - p(K_1) \leq K_2 - K_1 \,,$$
$$0 \leq P(K_2) - P(K_1) \leq K_2 - K_1 \,.$$

$\Rightarrow$ Roughly, the slopes of $p(K)$ and $P(K)$ are always between 0 and 1.

Options - Convexity

Proposition (Call Options - Convexity)

Both $c(K)$ *and* $C(K)$ *are* convex *functions of* K. *That is, for* $K_1, K_2 \geq 0$ *and* $\lambda \in [0,1]$,

$$c(\lambda K_1 + (1-\lambda)K_2) \leq \lambda c(K_1) + (1-\lambda)c(K_2)\,,$$
$$C(\lambda K_1 + (1-\lambda)K_2) \leq \lambda C(K_1) + (1-\lambda)C(K_2)\,.$$

Proposition (Put Options - Convexity)

Both $p(K)$ *and* $P(K)$ *are* convex *functions of* K. *That is, for* $K_1, K_2 \geq 0$ *and* $\lambda \in [0,1]$,

$$p(\lambda K_1 + (1-\lambda)K_2) \leq \lambda p(K_1) + (1-\lambda)p(K_2)\,,$$
$$P(\lambda K_1 + (1-\lambda)K_2) \leq \lambda P(K_1) + (1-\lambda)P(K_2)\,.$$

Options - Convexity

Proof.

We will show the result for European calls. The proofs of the other results are similar (for American options, remember to cover the short position by exercising).

Let $\bar{K} = \lambda K_1 + (1-\lambda)K_2$. Suppose we long $\lambda c(K_1)$ and $(1-\lambda)c(K_2)$, and short $c(\bar{K})$. At time T, the payoff is

$$\begin{aligned} &\lambda(S_T - K_1)_+ + (1-\lambda)(S_T - K_2)_+ - (S_T - \bar{K})_+ \\ =& \begin{cases} 0 & \text{if } S_T \geq K_2 \\ \lambda(S_T - K_1) - (S_T - \bar{K}) & \text{if } \bar{K} \leq S_T < K_2 \\ \lambda(S_T - K_1) & \text{if } K_1 \leq S_T < \bar{K} \\ 0 & \text{if } S_T < K_1. \end{cases} \end{aligned}$$

It is straightforward to verify that $\lambda(S_T - K_1) - (S_T - \bar{K}) = (1-\lambda)(K_2 - S_T) \geq 0$. Hence, this portfolio must have non-negative price, which implies the result. □

Options - Convexity - An Example

Example

Suppose we observe 3 call option prices today on the same stock and with same maturity: $c(50) = 14$, $c(59) = 8.9$ and $c(65) = 5$. Can we build an arbitrage strategy?

- Let $K_1 = 50$ and $K_2 = 65$. Observe that $0.4K_1 + 0.6K_2 = 59$.
- Call option should satisfy $0.4c(50) + 0.6c(65) \geq c(59)$.
- However, $0.4c(50) + 0.6c(65) = 0.4 \times 14 + 0.6 \times 5 = 8.6 < 8.9 = c(59)$.

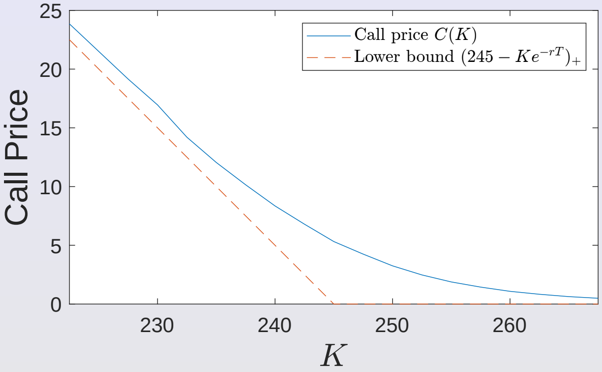
  $\Rightarrow$ There is arbitrage!

- We buy four $c(50)$, buy six $c(65)$, and sell ten $c(59)$. This gives an immediate cash flow of \$3.

  $\Rightarrow$ (Exercise) Verify that the terminal cash flow of this portfolio is always non-negative.

Options - Call Prices vs Strike

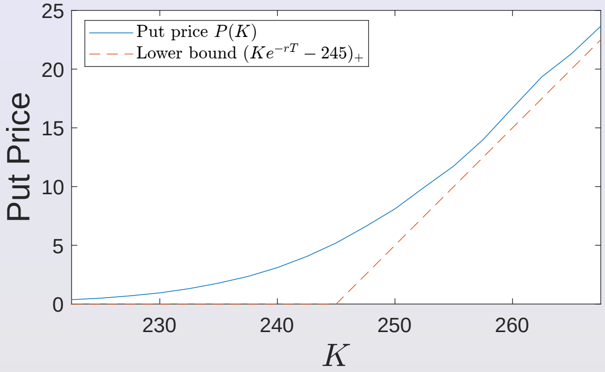
- The following shows the price of TSLA call options for different strikes. The spot price is $S_0 = \$245$.



- Note that call prices are decreasing, 1-Lipschitz, and convex.
- Deep in-the-money options seem to be slightly non-convex. This is due to bid-ask spreads (the plot shows ask prices).

Options - Put Prices vs Strike

- The following shows the price of TSLA put options for different strikes. The spot price is $S_0 = \$245$.



- Note that put prices are increasing, 1-Lipschitz, and convex.
- Again, due to bid-ask spreads, deep in-the-money options seem to be slightly non-convex.

Options - Different Expiry Dates

- We can also find relationships between options with different expiry dates.

Proposition (American Options - Different Expiry)

American options with longer expiry dates have higher prices. That is, let $T_1 \leq T_2$ be two expiration dates. Then for any time $t \leq T_1 \leq T_2$,

$$C(K, t, T_2) \geq C(K, t, T_1),$$
$$P(K, t, T_2) \geq P(K, t, T_1).$$

- This is because the longer American option lets you do everything the shorter option does.
- This does not hold for European options!

Options - Different Expiry Dates

- There is no clear relationship between the prices of European options with different expiries:

$$c(K, t, T_1) \text{ vs. } c(K, t, T_2)\,,$$
$$p(K, t, T_1) \text{ vs. } p(K, t, T_2)\,.$$

- For example, consider two European puts with strike K on the same stock, with expiration dates $T_1 = 1$ and $T_2 = 2$ years.
- Suppose that the company went out of business before one year has passed. Then at time 1, the first European put can be exercised for a payout of $\$K$.
- However, we need to wait until time 2 to exercise our other option and receive the payout of $\$K$. Hence, the put with more time to expiry is worth less in this case.

Options: Combinations and Strategies

Options - Combining Options

- When studying put-call parity, we showed how buying a call and shorting a put gives a similar payoff to a forward.
- Options can be combined in many different ways to engineer different payoffs.
  - Options are commonly used to provide insurance.
  - Options can also be used to speculate on volatility.

Options - Floors

- Suppose you are invested in a stock S since you think the price is going to go up.
- However, you may want to protect yourself against the potential downside.

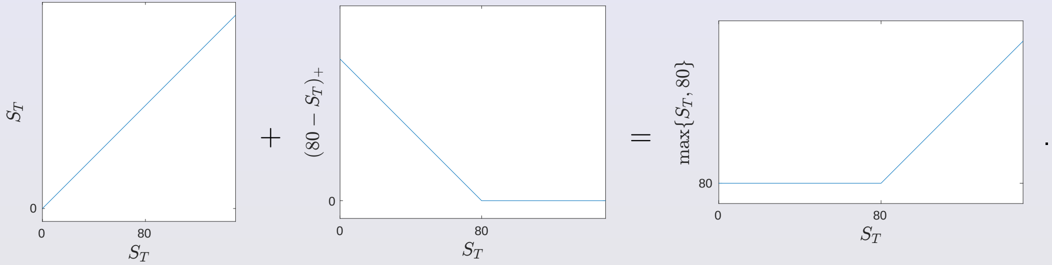
Definition (Floor)

A floor is when you buy a put option to cover the downside of being long an underlying. The resulting position consists of a long stock and a long put option.

- This guarantees that you can sell the asset for at least the strike price, providing a "floor" for your potential loss. Hence, a floor is a form of insurance.
- The payoff of this position is $S_T + (K - S_T)_+ = \max\{S_T, K\}$.

Options - Floors

- Graphically, by adding payoff diagrams, we have the following (for $K = \$80$):



Options - Covered Calls

- Now suppose that you want to give up some of the potential upside for a guaranteed payment. You can do this by selling a call on the stock.

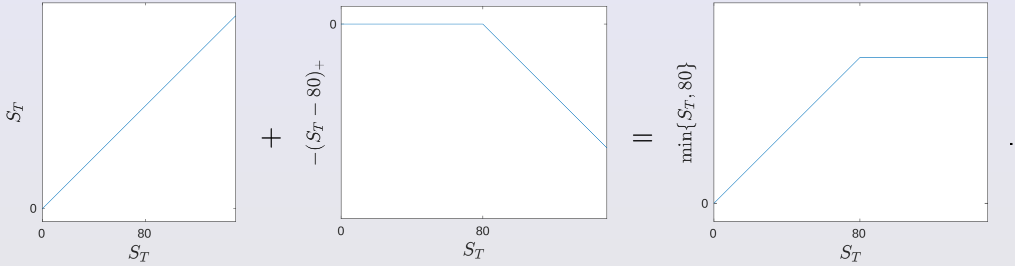
Definition (Covered Call)

A covered call is a position consisting of a long stock and a short call option on the stock.

- The payoff of this position is $S_T - (S_T - K)_+ = \min\{S_T, K\}$.
- By giving up the gains beyond K, you can obtain the option premium as a guaranteed payment.
- This is the principle behind covered call ETFs, which have been gaining popularity as ways to increase guaranteed income.

Options - Covered Calls

- Graphically, by adding payoff diagrams, we have the following:



Options - Caps

- Suppose you are short a stock S since you think the price is going to go down.
- Again, you may want to protect yourself against the potential downside of your short position.

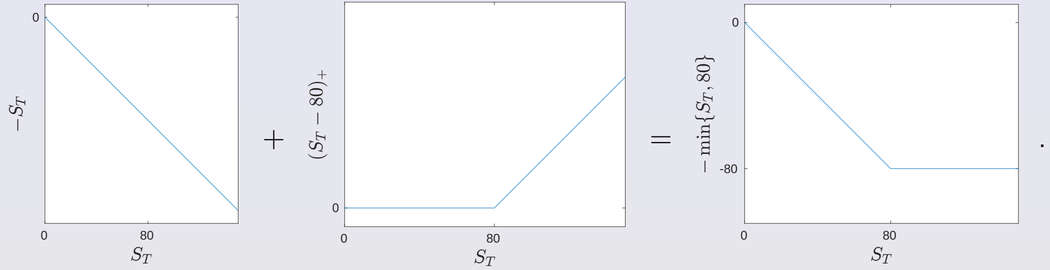
Definition (Cap)

A cap is when you buy a call option to cover the downside of a short position in the underlying. The resulting position consists of a short stock and a long call option.

- This guarantees that you can buy the asset for no more than the strike price to cover the short position. Hence, a cap is a form of insurance.
- The payoff of this position is $-S_T + (S_T - K)_+ = -\min\{S_T, K\}$.

Options - Caps

- Graphically, by adding payoff diagrams, we have the following:



Options - Covered Puts

- Now suppose that you want to give up some of the potential upside of your short position for a guaranteed payment. You can do this by selling a put on the stock.

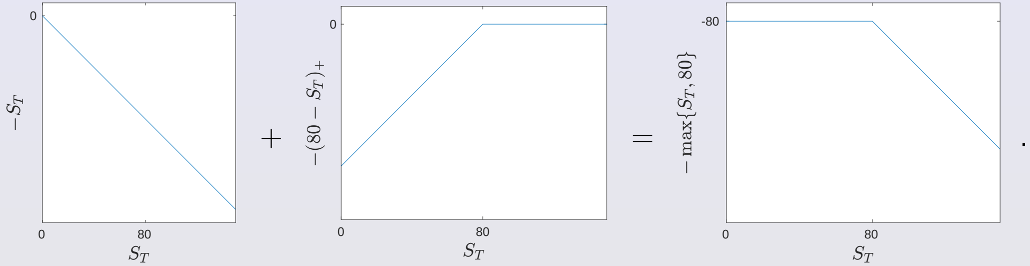
Definition (Covered Put)

A covered put is a position consisting of a short stock and a short put option on the stock.

- The payoff of this position is $-S_T - (K - S_T)_+ = -\max\{S_T, K\}$.
- By giving up the gains beyond K, you can obtain the option premium as a guaranteed payment.

Options - Covered Puts

- Graphically, by adding payoff diagrams, we have the following:



Options - Spreads

- Spreads are ways to bet on the movement of a stock using only calls or only puts.

Definition (Bull Spread)

A bull spread bets on an up movement by e.g. longing a call and shorting another call with a higher strike.

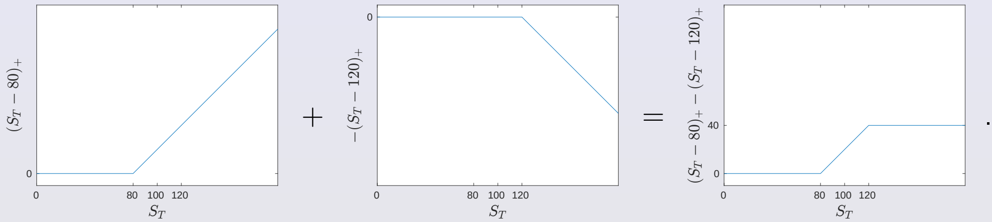
Definition (Bear Spread)

A bear spread bets on a down movement by e.g. longing a call and shorting another call with a lower strike.

- It can be shown (exercise) that spreads can also be constructed by using puts instead of calls.

Options - Bull Spread

- Suppose you long $c(80)$ and short $c(120)$. The payoff of this bull spread is the following:



$\Rightarrow$ Note that the payoff increases as the stock price increases.

Options - Collars

- Suppose you own a stock S that you think will go up.
- A collar combines both a floor and a covered call on this position. The result is a payoff that looks very similar to a bull spread.

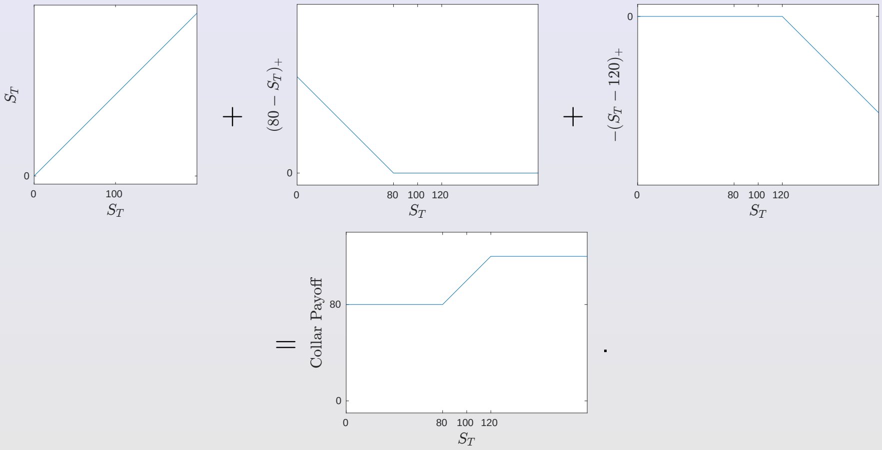
Definition (Collar)

A collar is a portfolio consisting of a long stock, a long put option with a lower strike, and a short call option with a higher strike.

- Bernie Madoff claimed that his portfolio used collars to lower volatility and get increased risk-adjusted returns! Of course, this is not possible because of no-arbitrage.

Options - Collars

- Suppose you are long a stock, long $p(80)$ and short $c(120)$. The payoff of this collar is the following:



Options - Straddles, Strangles, and Butterflies

- We can also use options to speculate on volatility.

Definition (Straddle)

A straddle is a strategy consisting of a long at-the-money call and a long at-the-money put.

Definition (Strangle)

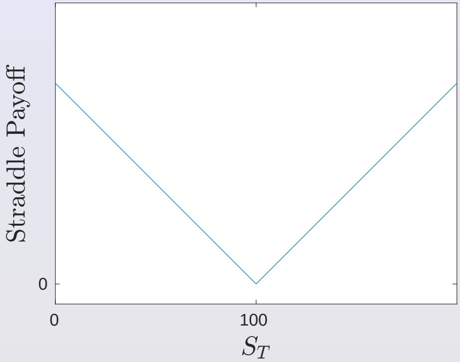
A strangle is a strategy consisting of a long out-of-the-money call and a long out-of-the-money put.

Definition (Butterfly)

A butterfly is a strategy consisting of a short straddle and a long strangle.

Options - Straddles, Strangles, and Butterflies

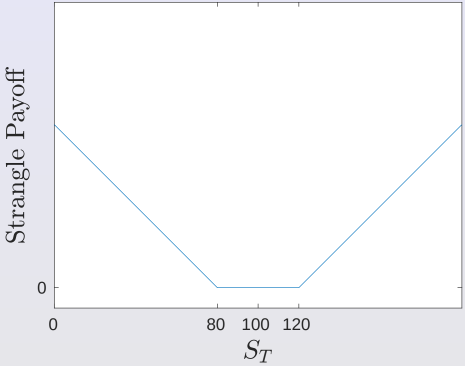
- Suppose a stock is currently trading at \$100. The payoff of a straddle on this stock is given by the following:



$\Rightarrow$ A straddle profits when the stock price deviates a lot from its current price. Hence, a straddle is long volatility.

Options - Straddles, Strangles, and Butterflies

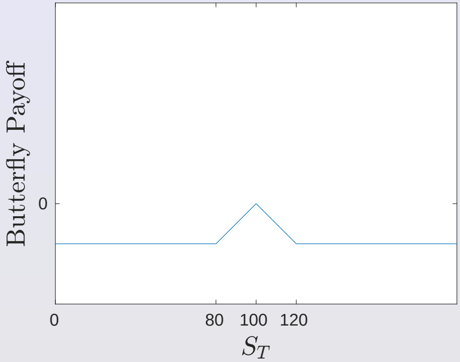
- Suppose a stock is currently trading at \$100. The payoff of a strangle on this stock (long $c(120)$, long $p(80)$) is given by the following:



⇒ A strangle profits when the stock price deviates a lot from its current price. Hence, a strangle is long volatility.

Options - Straddles, Strangles, and Butterflies

- The payoff of a butterfly on this stock, formed with the previous straddle and strangle, is given by the following:



$\Rightarrow$ A butterfly profits when the stock price does not deviate a lot from its current price. Hence, a butterfly is short volatility.

Part I - Summary

- We introduced the Principle of No Arbitrage and showed how to use it to prove results in a model-free setting.
- We defined forwards and futures, and learned how to find a fair future price.
- We defined call and put options, and obtained bounds on their prices.
    - We proved put-call parity.
- We showed how to combine different options to form different strategies.

What's Next

- Note that within a model free setting, we were ultimately unable to pinpoint an exact price of an option.
- In Part II, we will impose a model on the evolution of a stock. We start with models in discrete time.
- We will first show how to determine option prices in the binomial model.