Part II

Discrete-Time Models

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One-Period Binomial Model: Introduction

Stock Models

- In Part I, we obtained various results on option prices without any assumptions on the evolution of future stock prices $\{S_t\}_{t\geq 0}$.
  - Ultimately, we are unable to find an exact price of an option in a model-free setting.

Definition (Statistical Model)

A statistical model is a mathematical model consisting of a set of statistical assumptions for the purposes of generating sample data and making predictions.

- We will now look at different models of the stock price process $\{S_t\}_{t\geq 0}$.

$\Rightarrow$ In essence, we are going to introduce a probability measure $\mathbb{P}$ that describes the behaviour of stock prices.

One-Period Binomial Model - Introduction

- The simplest stock model is the one-period binomial model:

Definition (One-period Binomial Model)

Under the one-period binomial model, there is only one fixed time period, after which a stock can take either of two different values, each with positive probability.

- We will use the letter h to denote the length of this period.
- The single time period is the time interval between time $t = 0$ and $t = h$.
- At time $t = h$, the stock can take only one of two different values. Specifically, either:
  1. $S_h = uS_0$ with probability p_u, or
  2. $S_h = dS_0$ with probability $1 - p_u := p_d$. Note that $p_u, p_d \in (0, 1)$.

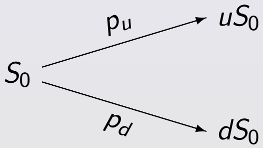
One-Period Binomial Model - Stock Price

- In other words,

$$\mathbb{P}(S_h = uS_0) = p_u, \text{ and } \mathbb{P}(S_h = dS_0) = p_d = 1 - p_u\,.$$

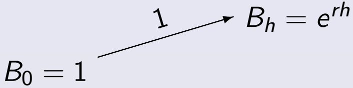
- We will assume that $u > d > 0$. Therefore u is the "upwards factor", and d is the "downwards factor".

- The evolution of the stock price over this time period is:



One-Period Binomial Model - Bond Price

- As before, we also assume that it is possible to invest at the risk-free rate r.
- Under the one-period binomial model, evolution of the value of a \$1 loan is:



- An investor can form portfolios consisting of stocks and bonds in fixed proportions. We now have a more precise definition of a portfolio:

Definition (Stock-Bond Portfolio)

A stock-bond portfolio is a pair of real numbers $\theta = (\Delta, b) \in \mathbb{R}^2$, which represents a position of Δ units of stock and b units of a bond (or alternatively, a \$$b$ loan).

One-Period Binomial Model - Arbitrage

- Let $\{V_t^\theta\}_{t\geq 0}$ denote the value process of a portfolio θ. A stock-bond portfolio $\theta = (\Delta, b)$ has value process $V_t^\theta = \Delta S_t + b \cdot b_t = \Delta S_t + be^{rt}$.
- Now that we have a probability measure $\mathbb{P}$, we have the following slightly refined definition of arbitrage:

Definition (Arbitrage Opportunity)

An arbitrage opportunity is a portfolio value process $\{V_t^\theta\}_{t\geq 0}$ such that:

(i) $V_0^\theta \leq 0$, and,

(ii) At the end of the period h, we have $\mathbb{P}(V_h^\theta \geq 0) = 1$ and $\mathbb{P}(V_h^\theta > 0) > 0$.

- Note that in this definition, there is arbitrage if we can make a (strictly positive) profit with positive probability.

One-Period Binomial Model - Arbitrage

- Again, we will take as an axiom the Principle of No Arbitrage:

Definition (Principle of No Arbitrage)

There are no arbitrage opportunities in this market.

- We can now start proving some results. As a first example, we have the following result, which essentially says that the bond cannot always be worse (or better) than the stock.

Proposition

The Principle of No Arbitrage holds in the one-period binomial model if and only if

$$d < e^{rh} < u\,.$$

One-Period Binomial Model - Arbitrage

Proof.

We will show that the Principle of No Arbitrage implies that $d < e^{rh} < u$. Suppose for the sake of contradiction that $e^{rh} \geq u$. We will show that this will yield an arbitrage opportunity.

Consider the stock-bond portfolio $\theta = (\Delta, b) = (-1, S_0)$. Then at time 0, we have

$$V_0^\theta = \Delta S_0 + be^{r \cdot 0} = -S_0 + S_0 = 0\,.$$

Note that in the one-period binomial model, we are only concerned with the future time h. At time h we have

$$V_h^\theta = \Delta S_h + be^{rh} = -S_h + S_0 e^{rh}\,.$$

One-Period Binomial Model - Arbitrage

Proof (cont'd).

With probability $p_u > 0$, we have

$$V_h^\theta = -uS_0 + S_0 e^{rh} = S_0(e^{rh} - u) \geq 0\,.$$

On the other hand, with probability $p_d > 0$, we have

$$V_h^\theta = -dS_0 + S_0 e^{rh} = S_0(e^{rh} - d) > S_0(e^{rh} - u) \geq 0\,.$$

Therefore we have found an arbitrage opportunity, since $\mathbb{P}(V_h \geq 0) = 1$ and $\mathbb{P}(V_h > 0) \geq p_d > 0$. We conclude that $e^{rh} < u$. As an exercise, it is possible to show the other inequality $d < e^{rh}$ in the same manner, by considering the portfolio $(1, -S_0)$.

For those interested in the converse (the "if" part in the "if and only if"), see Proposition 2.3 in Björk. □

One-Period Binomial Model - Risk-Neutral Measure

- Suppose that $d < e^{rh} < u$. Then e^{rh} is a convex combination of d and u. That is, there exists a number $q_u \in (0,1)$ such that

$$e^{rh} = q_u u + (1 - q_u) d \,.$$

- We will let $q_d = 1 - q_u$. The reason for this suspicious choice of notation will become clear very soon.
- Conversely, if there exists such a number $q_u \in (0,1)$, it is easy to see that $d < e^{rh} < u$. Hence, we have proven the following:

Lemma

We have $d < e^{rh} < u$ if and only if there exists some $q_u \in (0,1)$ such that

$$e^{rh} = q_u u + q_d d \,,$$

where $q_d = 1 - q_u$.

One-Period Binomial Model - Risk-Neutral Measure

- Now let's take the previous equation and multiply both sides by the current stock price S_0. Rearranging a little, we get:

$$
\begin{aligned}
e^{rh} &= q_u u + q_d d \\
S_0 e^{rh} &= q_u u S_0 + q_d d S_0 \\
S_0 &= e^{-rh}(q_u u S_0 + q_d d S_0)
\end{aligned}
$$

- Let $\mathbb{Q}$ be a probability measure, defined such that $\mathbb{Q}(S_h = uS_0) = q_u$ and $\mathbb{Q}(S_h = dS_0) = q_d$. Then the above expression simplifies to

$$
S_0 = e^{-rh}\mathbb{E}^{\mathbb{Q}}[S_h] .
$$

- In other words, the current stock price is the expectation of the future stock price under $\mathbb{Q}$, discounted at the risk-free rate.

One-Period Binomial Model - First Fundamental Theorem of Asset Pricing

- The measure $\mathbb{Q}$ we have constructed is important enough to warrant its own name.

Definition (Risk-Neutral Measure)

A risk-neutral measure is a probability measure $\mathbb{Q}$ such that

$$S_0 = e^{-rh}\mathbb{E}^{\mathbb{Q}}[S_h] \,.$$

- Over the past few slides, we have proven the following:

Theorem (First Fundamental Theorem of Asset Pricing (FTAP 1))

In the one-period binomial model, The Principle of No Arbitrage holds if and only if there exists a risk-neutral measure.

One-Period Binomial Model - Risk-Neutral Pricing

- The risk-neutral measure is important because we can use it to price derivatives.
- The payoff of a contingent claim at time h is a function of S_h. Let X denote the payoff. Then we have $X = \Phi(S_h)$ for some function Φ.
  - For example, for a call option, we have $\Phi(S_h) = \max\{S_h - K, 0\}$.
  - The function Φ is sometimes called the contract function.
- In the next section, we will prove the following, for the one-period binomial model:

Proposition (Risk-Neutral Pricing)

Let $\Pi_X(t)$ be the price of the derivative X at time t. Then

$$\Pi_X(0) = e^{-rh}\mathbb{E}^{\mathbb{Q}}[X] = e^{-rh}\mathbb{E}^{\mathbb{Q}}[\Phi(S_h)] .$$

One-Period Binomial Model - More on the Risk-Neutral Measure

- The risk-neutral measure is a "fake" measure, since it does not actually represent the probabilities of anything in the market.
  - In general, $\mathbb{P} \neq \mathbb{Q}$. That is, the risk-neutral measure is not the physical measure.
  - The risk-neutral measure is perhaps best interpreted as a mathematical tool to help us find prices of derivatives.
- The risk-neutral measure is called "risk-neutral" because of the pricing formula. If $\mathbb{P}$ were replaced by $\mathbb{Q}$, then the prices of everything are expectations discounted by the risk-free rate. The measure $\mathbb{Q}$ gets rid of all risk premiums in the market.
- Note that the pricing formula does not depend on $\mathbb{P}$! This is counter-intuitive, and something that takes some time to get used to.
  - One way to think about it is that changing p_u and p_d will change the probability of the outcomes of a contract X. However, it also changes the probability of the outcomes of the replicating portfolio too, and these cancel out.

One-Period Binomial Model - Risk-Neutral Probabilities

- From before, we saw that the risk-neutral probabilities in the one-period binomial model satisfy

$$e^{rh} = q_u u + q_d d \, .$$

- This is an equation with one unknown. Rearranging, we have the following formulas:

$$q_u = \frac{e^{rh} - d}{u - d} \, ,$$

$$q_d = \frac{u - e^{rh}}{u - d} \, .$$

One-Period Binomial Model: Risk-Neutral Pricing

One-Period Binomial Model - Completeness

- We will now prove the risk-neutral pricing formula. As before, we will construct replicating portfolios and apply the Law of One Price. The price of the replicating portfolio will be given precisely by the risk-neutral pricing formula.
- We first give some official definitions of these concepts that we saw in Part I.

Definition (Attainability)

A contingent claim $X = \Phi(S_h)$ is attainable if there exists a portfolio θ such that

$$V_h^{\theta} = X\,.$$

In this case, θ is a replicating portfolio for X.

Definition (Completeness)

A market is complete if it is arbitrage-free and all contingent claims are attainable.

One-Period Binomial Model - Completeness

Proposition

If the one-period binomial model is arbitrage-free, then it is complete.

Proof.

Let $X = \Phi(S_h)$ be the payoff of a derivative. We try to construct a replicating stock-bond portfolio $\theta^* := (\Delta^*, b^*)$. This is a replicating portfolio if its payoff $\Delta^* S_h + b^* e^{rh}$ matches that of X.

In the one-period binomial model, S_h is either uS_0 or dS_0, so X is either $\Phi(uS_0)$ or $\Phi(dS_0)$. Therefore if (Δ^*, b^*) is a replicating portfolio, it must solve

$$\Phi(uS_0) = \Delta^* uS_0 + b^* e^{rh}$$

$$\Phi(dS_0) = \Delta^* dS_0 + b^* e^{rh}.$$

One-Period Binomial Model - Completeness

Proof (cont'd).

This is a system of two linear equations and two unknowns. Solving for (Δ^*, b^*) gives

$$\Delta^* = \frac{\Phi(uS_0) - \Phi(dS_0)}{(u-d)S_0},$$

and

$$b^* = e^{-rh}\frac{u\Phi(dS_0) - d\Phi(uS_0)}{u-d}.$$

Therefore, every contingent claim is attainable, through the above replicating portfolio. □

Remark

We will see much later on that Δ^* is the delta of this derivative (i.e. the change in the derivative's price per change in stock price). This is why we use the letter Δ.

One-Period Binomial Model - Risk-Neutral Pricing

- We now prove the risk-neutral pricing formula for the one-period binomial model.

Proposition (Risk-Neutral Pricing, One-Period Binomial Model)

If the market is arbitrage free, then the current price $\Pi_X(0)$ *of a derivative with payoff* $X = \Phi(S_h)$ *at maturity* h *is*

$$\Pi_X(0) = e^{-rh}\mathbb{E}^{\mathbb{Q}}[X] = e^{-rh}\mathbb{E}^{\mathbb{Q}}[\Phi(S_h)] .$$

Proof.

We have seen that we can replicate the derivative with the portfolio $\theta^* = (\Delta^*, b^*)$ from the previous Proposition. By the Law of One Price, the derivative and the replicating portfolio must have the same price. That is,

$$\Pi_X(0) = V_0^{\theta^*} = \Delta^* S_0 + b^* .$$

One-Period Binomial Model - Risk-Neutral Pricing

Proof.

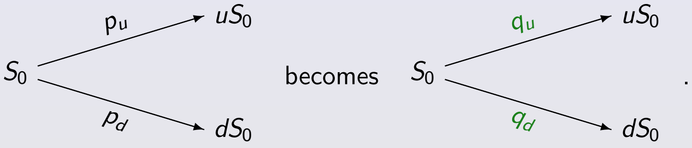
Plugging in the formulas for Δ^* and b^*, we have

$$\begin{aligned}
\Pi_X(0) = V_0^{\theta^*} &= \Delta^* S_0 + B^* \\
&= \frac{\Phi(uS_0) - \Phi(dS_0)}{(u-d)S_0} \times S_0 + e^{-rh}\frac{u\Phi(dS_0) - d\Phi(uS_0)}{u-d} \\
&= e^{-rh}\left[\frac{e^{rh}(\Phi(uS_0) - \Phi(dS_0))}{(u-d)} + \frac{u\Phi(dS_0) - d\Phi(uS_0)}{u-d}\right] \\
&= e^{-rh}\left[\Phi(uS_0)\left(\frac{e^{rh}-d}{u-d}\right) + \Phi(dS_0)\left(\frac{u-e^{rh}}{u-d}\right)\right] \\
&= e^{-rh}\left[\Phi(uS_0)q_u + \Phi(dS_0)q_d\right] \\
&= e^{-rh}\mathbb{E}^{\mathbb{Q}}[\Phi(S_h)] .
\end{aligned}$$

□

One-Period Binomial Model - Example

- Essentially, we have condensed the entire replicating portfolio argument into a single formula that is easy to use.
- In the following example, we will see how to compute prices using the following three steps:
  1. Replace the measure $\mathbb{P}$ with the risk-neutral measure $\mathbb{Q}$:



  2. Compute the expectation of the terminal payoff under $\mathbb{Q}$, $\mathbb{E}^{\mathbb{Q}}[\Phi(S_h)]$.
  3. Discount back to time 0 using the risk-free rate to get the price $e^{-rh}\mathbb{E}^{\mathbb{Q}}[\Phi(S_h)]$.

One-Period Binomial Model - Example

Example

Suppose a non-dividend paying stock follows a one-period binomial model with $S_0 = 100$, $u = 1.1$ and $d = 0.9$, and $h = 6$ months. Calculate the current price of a 6-month call option on this stock with strike price $K = 105$. Assume the risk-free rate is $r = 5\%$.

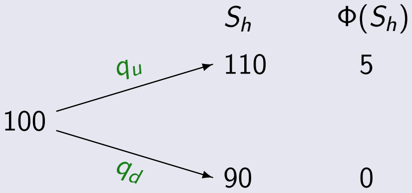
- We use the risk-neutral pricing formula. The first step is to find the risk-neutral measure by finding q_u and q_d. Using our formulas from before, we have

$$q_u = \frac{e^{rh} - d}{u - d} = \frac{e^{0.05 \times 0.5} - 0.9}{1.1 - 0.9} = 0.6266\,,$$
$$q_d = 1 - q_u = 0.3734\,.$$

One-Period Binomial Model - Example

Example

- Next, we find the expectation of the payoff of the call under the risk-neutral measure $\mathbb{Q}$. Under $\mathbb{Q}$, we have:



- Therefore we have

$$\mathbb{E}^{\mathbb{Q}}[\Phi(S_h)] = 5 \times q_u + 0 \times q_d = 5 \times 0.6266 = 3.1329\,,$$

$$\Pi_X(0) = e^{-rh}\mathbb{E}^{\mathbb{Q}}[\Phi(S_h)] = e^{-0.05\times 0.5} \times 3.1329 = \$3.06\,.$$

Multiperiod Binomial Model: Introduction

Multiperiod Binomial Model - Introduction

- The multiperiod binomial extends the one-period binomial model to multiple periods.
- Suppose we are given a fixed time horizon T.
  - We can split T into $k := T/h$ periods of length h.
  - Specifically, for each $t = 0, h, 2h, \ldots, T-h$, we assume that

    $$S_{t+h} = S_t Z_t \,,$$

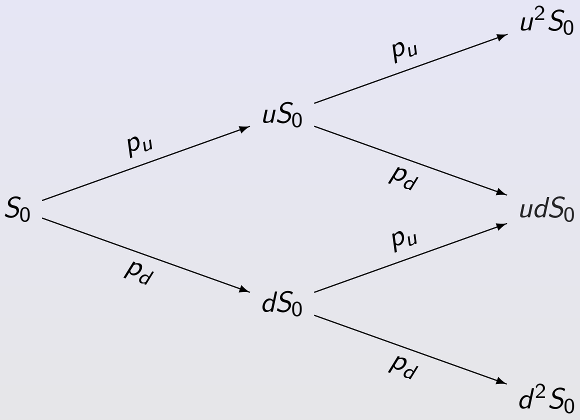
    where $Z_0, Z_h, \ldots, Z_{T-h}$ are i.i.d. random variables, with distribution

    $$Z = \begin{cases} u & \text{with probability } p_u \\ d & \text{with probability } p_d \end{cases} .$$

- In essence, this is a bunch of independent one-period models stuck together.

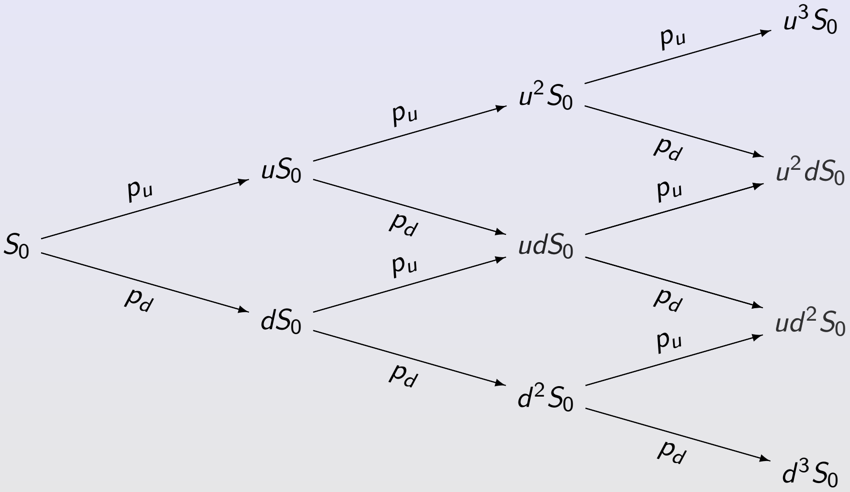
Multiperiod Binomial Model - Introduction

$k = 2$ periods:



Multiperiod Binomial Model - Introduction

$k = 3$ periods:



Multiperiod Binomial Model - Introduction

- And as before, we also have a risk-free asset that earns a continuously compounded interest rate r.
- The price of a \$1 loan over three periods is:

$$B_0 = 1 \xrightarrow{1} e^{rh} \xrightarrow{1} e^{2rh} \xrightarrow{1} e^{3rh} = e^{rT} = B_T$$

Multiperiod Binomial Model - Portfolios

- We see that as the number of periods increases, we can get more and more possible payoffs at time T. As we take $k \to \infty$, we can approximate S_T as if it were a continuous random variable.
- However, we will not be able to replicate all payoffs with a simple stock-bond portfolio held to time T.
- In order to replicate payoffs in the multiperiod model, we will need to adjust our portfolio over time. This is called rebalancing.
- We now need new definitions for portfolios and arbitrage, which encompass the possibility of rebalancing.

Multiperiod Binomial Model - Portfolios

Definition (Portfolio Strategy)

A portfolio strategy is a stochastic process $\theta = \{\theta_t\}_{t=0,h,\ldots,T}$ where:

- For each t, $\theta_t = (\Delta_t, b_t)$ is a function of $S_0, \ldots, S_{t-h}$.
- Δ_t denotes the number of shares of stock at time $t - h$ that are held until time t.
- b_t denotes the value of the risk-free asset at time $t - h$ that is held until time t.
- We set $\theta_0 = (\Delta_h, b_h e^{-rh})$ by convention.

$\Rightarrow$ We can think of a portfolio strategy as a series of stock-bond portfolios (one for each time period).

$\Rightarrow$ Note that a portfolio strategy is allowed to depend on the evolution of the stock so far.

Multiperiod Binomial Model - Portfolios

Definition (Value Process)

The value process of the portfolio strategy $\theta = \{\theta_t\}_{t=0,h,\ldots,T}$ is the stochastic process $\{V_t^\theta\}_{t=0,h,\ldots,T}$, where

$$V_t^\theta = \Delta_t S_t + b_t e^{rh} .$$

- That is, V_t^θ is the market value at time t of the portfolio θ.
- Note that by convention, $V_0^\theta = \Delta_h S_0 + b_h$.

Multiperiod Binomial Model - Portfolios

Definition (Self-Financing)

A portfolio strategy $\theta = \{\theta_t\}_{t=0,h,\ldots,T}$ is called self-financing if for all $t = h, 2h, \ldots, T-h$:

$$\underbrace{\Delta_t S_t + b_t e^{rh}}_{\text{Money in}} = \underbrace{\Delta_{t+h} S_t + b_{t+h}}_{\text{Money out}} .$$

$\Rightarrow$ If at time t we sell the portfolio, we get $\Delta_t S_t + b_t e^{rh}$. This is just enough money to buy the portfolio that we intend to hold for the next time period, which costs $\Delta_{t+h} S_t + b_{t+h}$.

$\Rightarrow$ Self-financing portfolios are usually what we are concerned with, since the price of a self-financing portfolio is its time-0 price. There are no withdrawals or contributions to the portfolio after time 0.

Multiperiod Binomial Model - Arbitrage

Definition (Arbitrage Opportunity)

An arbitrage opportunity is a self-financing portfolio strategy θ such that:

(i) $V_0^\theta \leq 0$, and,

(ii) At expiry T, we have $\mathbb{P}(V_T^\theta \geq 0) = 1$ and $\mathbb{P}(V_T^\theta > 0) > 0$.

- As always, we will take as an axiom the Principle of No Arbitrage.

Definition (Principle of No Arbitrage)

There are no arbitrage opportunities in this market.

Multiperiod Binomial Model - Arbitrage

- Similar to the one-period model, we have the following result. The proof is omitted, but one direction of the result should be obvious.

Proposition

The Principle of No Arbitrage holds in the multiperiod model if and only if

$$d < e^{rh} < u\,.$$

- Again, this inequality is true if and only if e^{rh} is a convex combination of u and d. So we can define q_u, q_d as before...

Multiperiod Binomial Model - Risk-Neutral Measure

Definition (Risk-Neutral Measure)

A risk-neutral measure is a probability measure $\mathbb{Q}$ such that for each $t = 0, h, \ldots, T - h$,

$$S_t = e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[S_{t+h} \,\middle|\, S_0, \ldots, S_t\right] = e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[S_{t+h} \,\middle|\, S_t\right]$$

- We will verify that the measure $\mathbb{Q}$ under which $Z_0, Z_h, \ldots, Z_{T-h}$ are i.i.d. with distribution

$$Z = \begin{cases} u & \text{with probability } q_u \\ d & \text{with probability } q_d \end{cases},$$

is indeed a risk-neutral measure.

Multiperiod Binomial Model - Risk-Neutral Measure

- For any time $t = 0, h, \ldots, T - h$ and any possible stock price s,

$$
\begin{aligned}
e^{-rh}\mathbb{E}^{\mathbb{Q}}[S_{t+h}|S_t = s] &= e^{-rh}\mathbb{E}^{\mathbb{Q}}[S_t Z_t|S_t = s] \\
&= se^{-rh}\mathbb{E}^{\mathbb{Q}}[Z_t|S_t = s] \\
&= se^{-rh}\mathbb{E}^{\mathbb{Q}}[Z_t] \\
&= se^{-rh}(\underbrace{uq_u + dq_d}_{e^{rh}}) \\
&= s\,.
\end{aligned}
$$

Multiperiod Binomial Model - Risk-Neutral Measure

- We have proven the following:

Theorem (First Fundamental Theorem of Asset Pricing (FTAP 1))

In the multiperiod binomial model, The Principle of No Arbitrage holds if and only if there exists a risk-neutral measure.

- As before, the real strength of the risk-neutral measure is that it will help us price derivatives, by giving us a concise formula that we can use. This is the focus of the following section.

Multiperiod Binomial Model: Risk-Neutral Valuation

Multiperiod Binomial Model - Completeness

- Similar to the one-period binomial model, we can show that every contingent claim can be replicated by a self-financing portfolio. First, some definitions of familiar concepts in this context:

Definition (Contingent Claim)

A contingent claim is a random variable $X = \Phi(S_0, S_h, \ldots, S_T)$, where Φ is a deterministic contract function. The interpretation is that X denotes the payoff of a derivative at the terminal time T.

Definition (Attainability)

A contingent claim $X = \Phi(S_0, S_h, \ldots, S_T)$ is attainable if there exists a self-financing portfolio strategy θ^* such that

$$V_T^{\theta^*} = X \,.$$

Multiperiod Binomial Model - Completeness

Definition (Completeness)

A market is complete if it is arbitrage-free and all contingent claims are attainable.

Proposition

If the multiperiod binomial model is arbitrage-free, then it is complete.

Multiperiod Binomial Model - Completeness

Proof (Sketch).

A formal proof of this would require induction and would be extremely tedious.

Instead, we can see that this is true by going backwards along the tree. The final time period is essentially a one-period binomial model, which we know is complete and can be replicated at time $T - h$. If the cost of this replicating portfolio matches the market value of this replicating portfolio, then we can rebalance into the replicating portfolio and still satisfy the self-financing condition.

Hence, we just need that the market value of our portfolio at $T - h$ matches the value of the replicating portfolio. We can then shrink the tree by removing the final period. Repeating this process until time 0 gives the desired self-financing portfolio. □

Multiperiod Binomial Model - Price Processes

- We can now conclude using the Law of One Price that the price of a derivative X must be the same as the price of the replicating portfolio strategy θ^*:

$$\Pi_X(0) = V_0^{\theta^*} .$$

- However, we can actually do something better in this case: we can apply the Law of One Price at every node of the tree, not just the root! Doing so proves the following:

Proposition

If θ^* is a replicating portfolio strategy for X, then for every $t = 0, h, 2h, \ldots, T$,

$$\Pi_X(t) = V_t^{\theta^*} .$$

Multiperiod Binomial Model - Price Processes

- Note that $V_t^{\theta^*}$ is a function of the stock prices up to time t. By the previous result, this means that $\Pi_X(t)$ is also a function of stock prices up to time t.
- In other words, if we know what node we are at, we know what the value of $\Pi_X(t)$ is.
- This property is important enough to warrant its own definition:

Definition (Adapted Process)

A stochastic process $\{Y_t\}_{t\geq 0}$ is adapted (to $\{S_t\}_{t\geq 0}$) if for each time j, the value of Y_j is determined by $\{S_t\}_{j\geq t\geq 0}$.

- Therefore, the price process $\Pi_X(t)$ is an adapted process.

Multiperiod Binomial Model - Recursive Valuation

- Finally, we can find a formula for the price process $\Pi_X(t)$ at every node.
- In the one-period case, we found the portfolio weights explicitly, but the formulas were fairly complicated. This time, we will use some tools from statistics.
- First, we have the following result, which is the analogue of the risk-neutral valuation formula for the one-period model.

Proposition

Let $\mathbb{Q}$ be the risk-neutral probability measure. For every self-financing portfolio strategy θ and any time $t = 0, h, \ldots, T - h$, we have

$$V_t^\theta = e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[V_{t+h}^\theta \,\middle|\, S_0, \ldots, S_t\right].$$

Multiperiod Binomial Model - Recursive Valuation

Proof.

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[V^{\theta}_{t+h} \,\middle|\, S_0, \ldots, S_t\right] &= \mathbb{E}^{\mathbb{Q}}\left[\Delta_{t+h} S_{t+h} + b_{t+h} e^{rh} \,\middle|\, S_0, \ldots, S_t\right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[\Delta_{t+h} S_{t+h} \,\middle|\, S_0, \ldots, S_t\right] + \mathbb{E}^{\mathbb{Q}}\left[b_{t+h} e^{rh} \,\middle|\, S_0, \ldots, S_t\right] \\
&= \Delta_{t+h} \mathbb{E}^{\mathbb{Q}}\left[S_{t+h} \,\middle|\, S_0, \ldots, S_t\right] + b_{t+h} e^{rh} \\
&= \Delta_{t+h} e^{rh} S_t + b_{t+h} e^{rh} \qquad \text{(definition of risk-neutral measure)} \\
&= e^{rh}(\Delta_{t+h} S_t + b_{t+h}) \\
&= e^{rh}(\Delta_t S_t + b_t e^{rh}) \qquad \text{(self-financing)} \\
&= e^{rh} V^{\theta}_t .
\end{aligned}
$$

□

Multiperiod Binomial Model - Recursive Valuation

- As a corollary, we have the following:

Proposition (Recursive Valuation)

Let $\mathbb{Q}$ be the risk-neutral probability measure, and suppose that X is a contingent claim. Then at any time $t = 0, h, \ldots, T - h$, we have

$$\Pi_X(t) = e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[\Pi_X(t+h) \,|\, S_0, S_h, \ldots, S_t\right].$$

Proof.

Since the multiperiod model is complete, there is a replicating portfolio θ^* for X. The result follows immediately from the previous Proposition and the fact that $\Pi_X(t) = V_t^{\theta^*}$ for every time $t = 0, h, \ldots, T$. □

Multiperiod Binomial Model - Risk-Neutral Valuation

- Using recursive valuation, we can prove the most important result in Part II:

Theorem (Risk-Neutral Valuation)

Suppose a multiperiod binomial model is arbitrage free. Then the risk-neutral price of a contingent claim X at time $t = 0, h, \ldots, T$ is given by

$$\Pi_X(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[X \mid S_0, S_h, \ldots, S_t\right].$$

In particular,

$$\Pi_X(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[X].$$

Multiperiod Binomial Model - Risk-Neutral Valuation

Proof.

We use induction, backward in time.

Base case $(t = T)$: At time T, the price must match the payoff. Recall that X denotes the time-T payoff of the derivative, which is of the form $\Phi(S_0, S_h, \ldots, S_T)$. Hence, given the stock prices up until time T, we know that the value of X is. Therefore we have

$$\Pi_X(T) = X = \mathbb{E}^{\mathbb{Q}}[X \mid S_0, \ldots, S_T] .$$

Multiperiod Binomial Model - Risk-Neutral Valuation

Proof (cont'd).

Inductive step: Now suppose the result holds for time $t+h$. Then we want to show that the result holds for time t. By recursive valuation, we have:

$$\begin{aligned}
\Pi_X(t) &= e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[\Pi_X(t+h) \,\middle|\, S_0, S_h, \ldots, S_t\right] \\
&= e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-(t+h))}\mathbb{E}^{\mathbb{Q}}\left[X \,\middle|\, S_0, S_h, \ldots, S_{t+h}\right] \,\middle|\, S_0, S_h, \ldots, S_t\right] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[X \,\middle|\, S_0, S_h, \ldots, S_{t+h}\right] \,\middle|\, S_0, S_h, \ldots, S_t\right] \\
&= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[X \,\middle|\, S_0, S_h, \ldots, S_t\right],
\end{aligned}$$

where we apply the tower rule in the last line. This concludes the proof. □

Option Pricing in the Binomial Model: European Options

Binomial Model - Option Pricing

- We saw two important results:

Recursive Valuation

Let $\mathbb{Q}$ be the risk-neutral probability measure, and suppose that X is a contingent claim. Then at any time $t = 0, h, \ldots, T - h$, we have

$$\Pi_X(t) = e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[\Pi_X(t+h) \mid S_0, S_h, \ldots, S_t\right].$$

Risk-Neutral Valuation

The risk-neutral price of a contingent claim X at time $t = 0, h, \ldots, T$ is given by

$$\Pi_X(t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[X \,\middle|\, S_0, S_h, \ldots, S_t\right].$$

- We will now see how to use these formulas to price options in the binomial model. We start with a European put.

Binomial Model - Recursive Valuation

Example (European Put Option)

Suppose we are given the following information on a European put option:

- The strike is $K = 160$.
- The time to expiry is $T = 6$ months.
- The spot price is $S_0 = 150$.
- We use a two-period model. The stock can either increase by 30% or decrease by 30% after one period.
- The risk free rate is $r = 6\%$.
- The stock does not pay dividends.

What should the price of the option be?

Binomial Model - Recursive Valuation

Example (European Put Option)

- Note that the physical probabilities p_u, p_d are not given. They are not needed to price!
- We have two periods of length $h = T/2 = 3$ months.
- From the information, we conclude that $u = 1.3$ and $d = 0.7$.
- Then using the formula for risk-neutral probabilities, we have

$$q_u = \frac{e^{rh} - d}{u - d} = \frac{e^{0.06 \times 3/12} - 0.7}{1.3 - 0.7} = 0.5252\,,$$

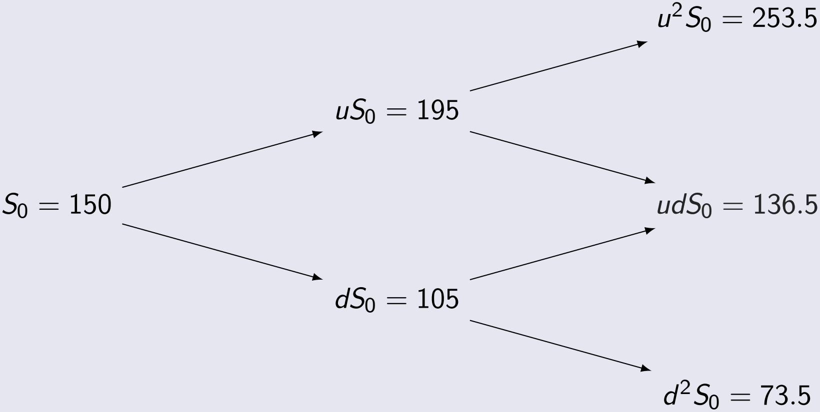
and

$$q_d = \frac{u - e^{rh}}{u - d} = 1 - q_u = 0.4748\,.$$

Binomial Model - Recursive Valuation

Example (European Put Option)

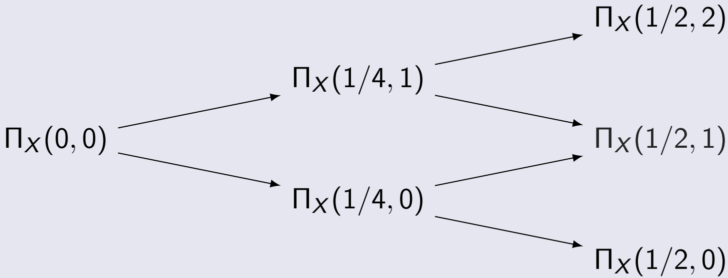
The two-period tree showing the evolution of the stock price is the following:



Binomial Model - Recursive Valuation

Example (European Put Option)

We now wish to find the price of the option. We first do this by using the recursive formula to fill out the values of the following tree:

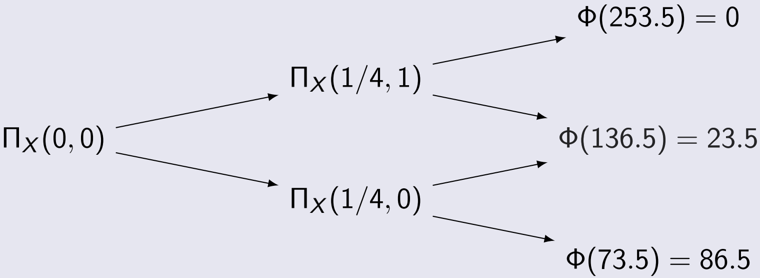


Remember that $\Pi_X(t)$ depends on the stock prices up until time t, so the values are different at every node. Here, the notation $\Pi_X(t,j)$ means the option price at time t, given j up-movements of the stock.

Binomial Model - Recursive Valuation

Example (European Put Option)

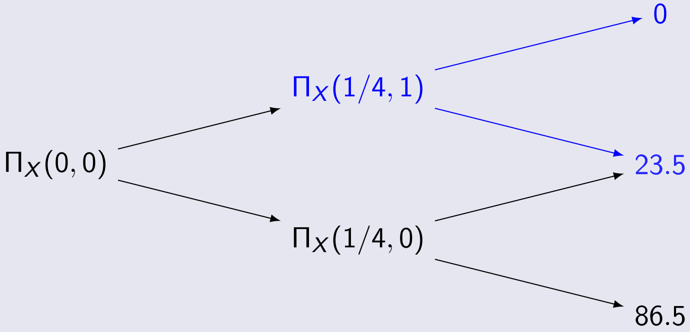
The payoff of the option at maturity is $\Phi(S_T) = (K_2 - S_T)_+$. We can replace the rightmost nodes with their values:



Binomial Model - Recursive Valuation

Example (European Put Option)

Let's first examine the upper node, which corresponds to the case where $S_1 = uS_0$.

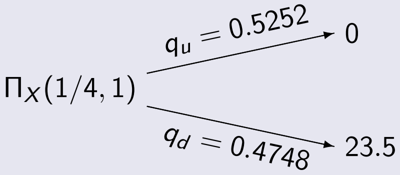


Binomial Model - Recursive Valuation

Example (European Put Option)

In this case, the recursive valuation formula gives

$$\Pi_X(1/4, 1) = e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[\Pi_X(1/2) \,\middle|\, S_1 = uS_0\right].$$



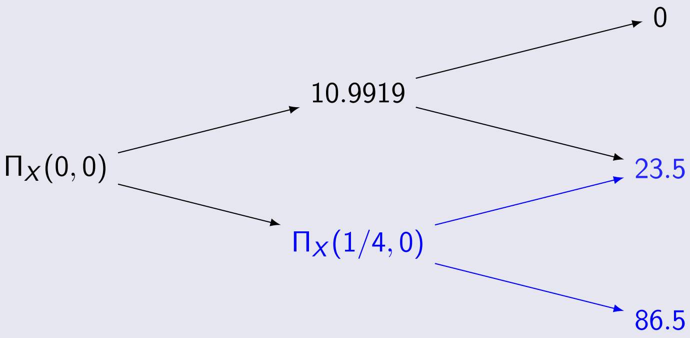
Therefore,

$$\Pi_X(1/4, 1) = e^{-0.06\times 3/12}(0.5252 \times 0 + 0.4748 \times 23.5) = 10.9919\,.$$

Binomial Model - Recursive Valuation

Example (European Put Option)

Now we do the same thing to the bottom node, which corresponds to the case where $S_1 = dS_0$.

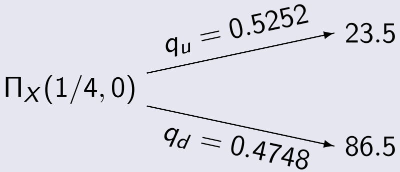


Binomial Model - Recursive Valuation

Example (European Put Option)

In this case, the recursive valuation formula gives

$$\Pi_X(1/4, 0) = e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[\Pi_X(1/2) \,\middle|\, S_1 = dS_0\right].$$



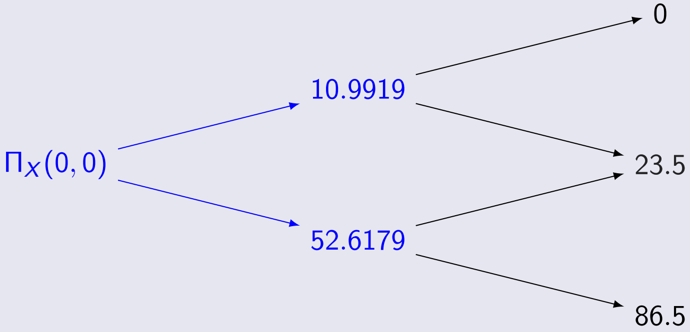
Therefore,

$$\Pi_X(1/4, 0) = e^{-0.06\times 3/12}(0.5252 \times 23.5 + 0.4748 \times 86.5) = 52.6179\,.$$

Binomial Model - Recursive Valuation

Example (European Put Option)

Finally we can find the price at time 0 using the values we already obtained.

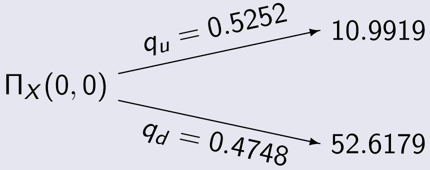


Binomial Model - Recursive Valuation

Example (European Put Option)

The recursive valuation formula gives

$$\Pi_X(0,0) = e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[\Pi_X(1/4) \,\middle|\, S_0\right] = e^{-rh}\mathbb{E}^{\mathbb{Q}}\left[\Pi_X(1/4)\right].$$



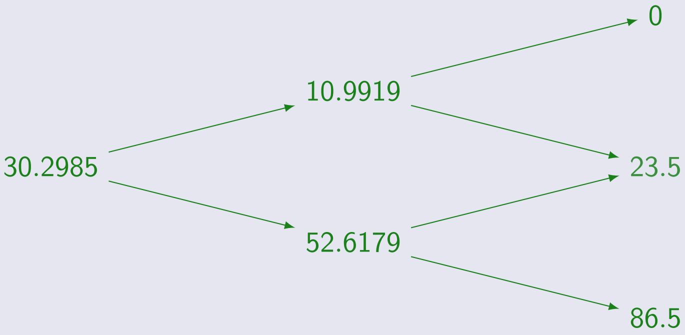
Therefore,

$$\Pi_X(0,0) = e^{-0.06\times 3/12}(0.5252 \times 10.9919 + 0.4748 \times 52.6179) = 30.2985\,.$$

Binomial Model - Recursive Valuation

Example (European Put Option)

We conclude that the price of the option is \$30.30. The price at every node is now known.



Binomial Model - Risk-Neutral Valuation

Example (European Put Option)

- We can also use the risk-neutral valuation formula as a shortcut. Note that the option price at time zero is

$$\Pi_X(0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(K - S_2)_+] \, .$$

- S_2 can take 3 values, based on the number of up-movements j:

$$S_2 = S_0 u^j d^{2-j} \, , \quad j = 0, 1, 2 \, .$$

- Looking at the tree, there are $\binom{2}{j}$ ways for $S_2 = S_0 u^j d^{2-j}$. Therefore, we have

$$\mathbb{Q}\left[S_2 = S_0 u^j d^{2-j}\right] = \binom{2}{j} q_u^j q_d^{2-j} \, .$$

- Note that these probabilities come from the binomial distribution.

Binomial Model - Risk-Neutral Valuation

Example (European Put Option)

Hence, we have

$$
\begin{aligned}
\Pi_X(0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\big[(K - S_2)_+\big] \\
&= e^{-rT}\sum_{j=0}^{2}\binom{2}{j} q_u^j q_d^{2-j}(K - S_0 u^j d^{2-j})_+ \\
&= e^{-0.06\times 6/12}\big[q_u^2(K - S_0 u^2)_+ + 2q_u q_d(K - S_0 ud)_+ + q_d^2(K - S_0 d^2)_+\big] \\
&= e^{-0.03}\big[(0.5252)^2(160 - 253.5)_+ + 2(0.5252)(0.4748)(160 - 136.5)_+ \\
&\qquad\quad + (0.4748)^2(160 - 73.5)_+\big] \\
&= 30.2985\,.
\end{aligned}
$$

Binomial Model - Risk-Neutral Valuation

- In general, if the model has k periods, then S_T can take $k+1$ different values based on the number of up-movements j:

$$S_T = S_0 u^j d^{k-j}\,, \quad j = 0, 1, 2, \ldots, k\,.$$

- There are $\binom{k}{j}$ different ways we can have j up-movements. The probability of each of these ways, under $\mathbb{Q}$, is $q_u^j q_d^{k-j}$.
- Therefore,

$$\mathbb{Q}\big[S_T = S_0 u^j d^{k-j}\big] = \binom{k}{j} q_u^j q_d^{k-j}\,.$$

Binomial Model - Risk-Neutral Valuation

- Therefore, for a contingent claim with a payoff of the form $X = \Phi(S_T)$, we have the following formula:

$$\begin{aligned}\Pi_X(0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Phi(S_T)\right] \\ &= e^{-rT}\sum_{j=0}^{k}\binom{k}{j} q_u^j q_d^{2-j}\Phi(S_0 u^j d^{2-j}).\end{aligned}$$

- An advantage of this formula is that it is very easy to implement in code. There are only $O(k)$ terms to calculate, whereas the recursive formula would require $O(k^2)$ terms.

Option Pricing in the Binomial Model: American Options

Binomial Model - American Options

- We can also use a binomial tree to price American options. Recall that the option can be exercised at any point before expiration, which makes writing explicit payoff functions difficult.
- However, with a stock model in place, we are actually able to determine how to optimally exercise American options!
- We will assume that exercise is only possible at the end of each time period.
- In this section, we will be a little less rigorous. For those who are interested, this is called an optimal stopping problem, and is solved using dynamic programming.

Binomial Model - American Options

- Denote by (t, j) the node at time t with j up-movements. At any node, we can either exercise or hold the option. How do we decide what to do?
- We decide by comparing two values: the exercise value and the continuation value.
    - The exercise value is the value of the option if it is exercised:

$$E(t,j) = \Phi(S(t,j)) = \Phi(S_0 u^j d^{t/h-j}) .$$

    - The continuation value is the value of the option if we were to hold, and is given recursively:

$$H(t,j) = e^{-rh}\left[q_u \Pi(t+h, j+1) + q_d \Pi(t+h, j)\right] .$$

    - We exercise the option when the exercise value is greater:

$$\Pi(t,j) = \max\{E(t,j), H(t,j)\} .$$

Binomial Model - American Options

- We can calculate the values of Π recursively since we know the value of the option at maturity:
$$\Pi(T, j) = \Phi(S(T, j)) = \Phi(S_0 u^j d^{k-j}) \,.$$
- By repeating this process, we can find the current price of the American option $\Pi(0, 0)$.
- Note that we have to use recursion here, and there is no shorter formula. This is because we need to check the possibility of exercise at every node.
- Essentially, we recursively calculate option prices at every node, but if the exercise value is greater, we replace the option price with its exercise value. In the following, we re-examine our put option example.

Binomial Model - Recursive Valuation of American Options

Example (American Put Option)

Suppose we are given the following information on an American put option:

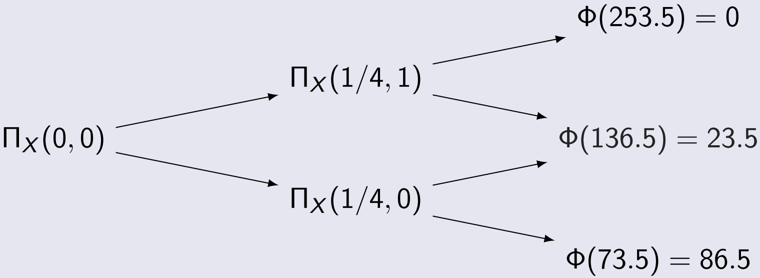
- The strike is $K = 160$.
- The time to expiry is $T = 6$ months.
- The spot price is $S_0 = 150$.
- We use a two-period model. The stock can either increase by 30% or decrease by 30% after one period.
- The risk free rate is $r = 6\%$.
- The stock does not pay dividends.

What should the price of the option be?

Binomial Model - Recursive Valuation of American Options

Example (American Put Option)

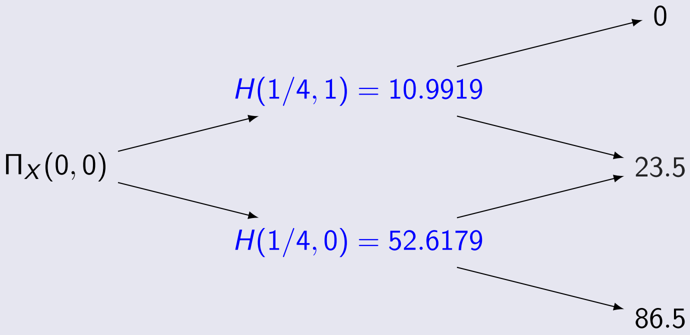
As before, the value of the option at maturity is the payoff:



Binomial Model - Recursive Valuation of American Options

Example (American Put Option)

Using the same formula from before, we can find the continuation values at the middle nodes:



These are not the values of Π_X, since we need to consider exercise values.

Binomial Model - Recursive Valuation of American Options

Example (American Put Option)

Comparing the continuation values to the exercise values, we have:

- At node $(1/4, 1)$:

$$E(1/4,1) = \Phi(S_0 u) = (160 - 195)_+ = 0\,,$$
$$\Pi(1/4,1) = \max\{E(1/4,1), H(1/4,1)\} = \max\{0, 10.9919\} = 10.9919\,.$$

- At node $(1/4, 0)$:

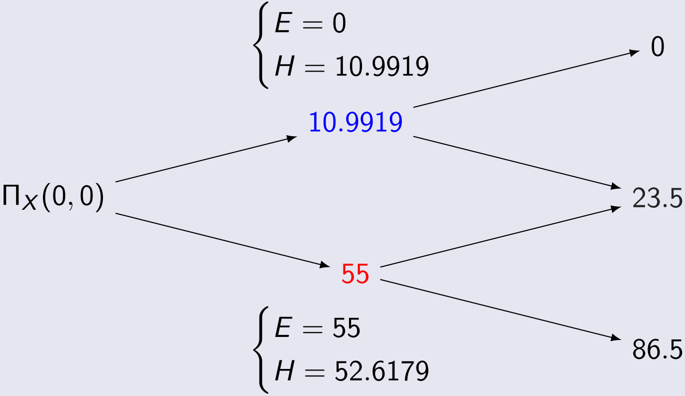
$$E(1/4,0) = \Phi(S_0 d) = (160 - 105)_+ = 55\,,$$
$$\Pi(1/4,0) = \max\{E(1/4,0), H(1/4,0)\} = \max\{55, 52.6179\} = 55\,.$$

This implies that the option will be exercised if the stock goes down in the first period!

Binomial Model - Recursive Valuation of American Options

Example (American Put Option)

Hence, we have the following:



Binomial Model - Recursive Valuation of American Options

Example (American Put Option)

Repeating this process for node $(0,0)$, we have:

$$\begin{aligned} H(0,0) &= e^{-rh}(q_u \Pi(1/4,1) + q_d \Pi(1/4,0)) \\ &= e^{-rh}(q_u \times 10.9919 + q_d \times 55) = 31.4127\,, \\ E(0,0) &= \Phi(S_0) = (160 - 150)_+ = 10\,, \\ \Pi(0,0) &= \max\{E(0,0), H(0,0)\} = \max\{10, 31.4127\} = 31.4127\,. \end{aligned}$$

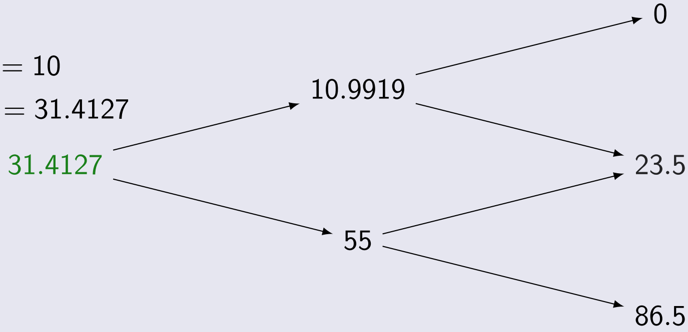
Hence, the price of this American put option is \$31.41. Note that this price is greater than that of the corresponding European put \$30.30.

Binomial Model - Recursive Valuation of American Options

Example (American Put Option)

Hence, we have the following:

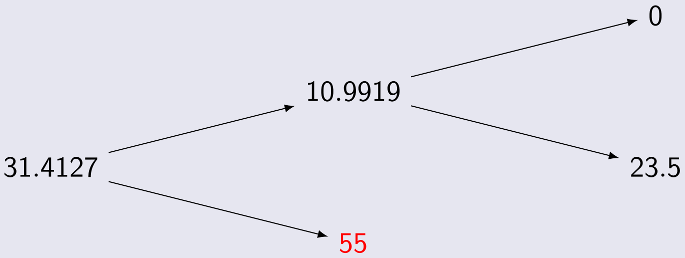
$$\begin{cases} E = 10 \\ H = 31.4127 \end{cases}$$



Binomial Model - Recursive Valuation of American Options

Example (American Put Option)

Since we always exercise after a down movement in period 1, the actual value of the option is:



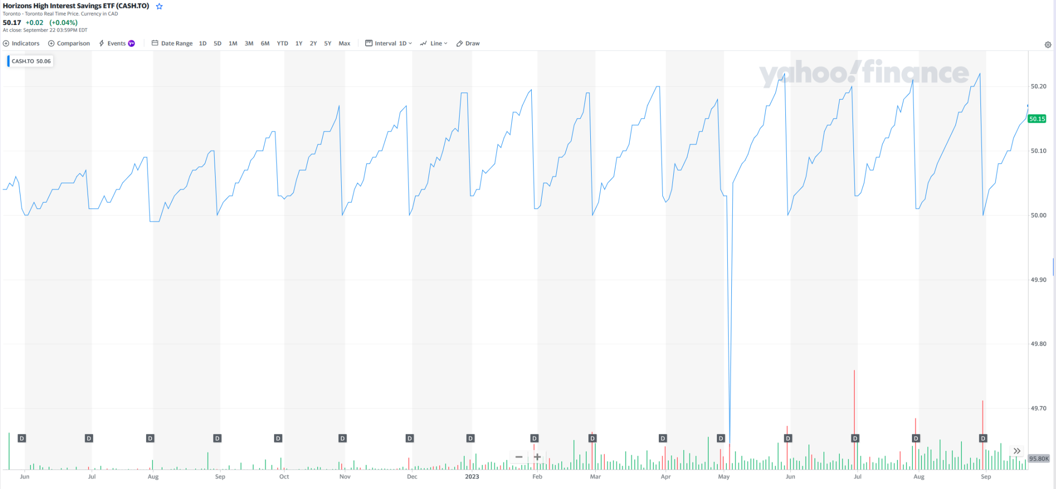
Option Pricing in the Binomial Model: Dividends

Binomial Model - Dividends

- So far we have only considered situations where the stock does not pay dividends.
- We can also use binomial trees to price options on dividend-paying stock. However, it is important to remember the following:
  - The holder of an option is not entitled to dividends. Only shareholders receive dividends.
  - The payment of dividends affects the price of the stock. The underlying stock price drops by the amount of the dividend when a dividend is paid.
- The price of a stock right before a dividend payment is called the cum-dividend price.
- The price of a stock right after a dividend payment is called the ex-dividend price.
- The difference between the two is the amount of dividend paid.

Binomial Model - Dividends

- Here is a chart for CASH.TO, a money market ETF:[1]



[1] Market data as of EOD Sept. 22, 2023. CASH.TO is the Horizons High Interest Savings ETF, which currently yields 5.39%.

Binomial Model - Continuous Dividends

- For a continuous dividend rate of δ, we have a convenient shortcut. We can do exactly the same thing as the no-dividend case, but using

$$q_u = \frac{e^{(r-\delta)h} - d}{u-d}, \qquad q_d = \frac{u - e^{(r-\delta)h}}{u-d}.$$

- In this case, it is possible to verify that the Principle of No Arbitrage holds if and only if

$$d < e^{(r-\delta)h} < u.$$

- Redoing everything will eventually lead us to the formula...

Binomial Model - Continuous Dividends

Example (European Put Option, Continuous Dividends)

Suppose we are given the following information on a European put option:

- The strike is $K = 160$.
- The time to expiry is $T = 6$ months.
- The spot price is $S_0 = 150$.
- We use a two-period model. The stock can either increase by 30% or decrease by 30% after one period.
- The risk free rate is $r = 6\%$.
- The stock pays a continuous dividend at a rate of $\delta = 4\%$.

What should the price of the option be?

Binomial Model - Continuous Dividends

Example (European Put Option, Continuous Dividends)

In this case, using the new formula for risk-neutral probabilities, we have

$$q_u = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(0.06-0.04)\times 3/12} - 0.7}{1.3 - 0.7} = 0.5084\,,$$

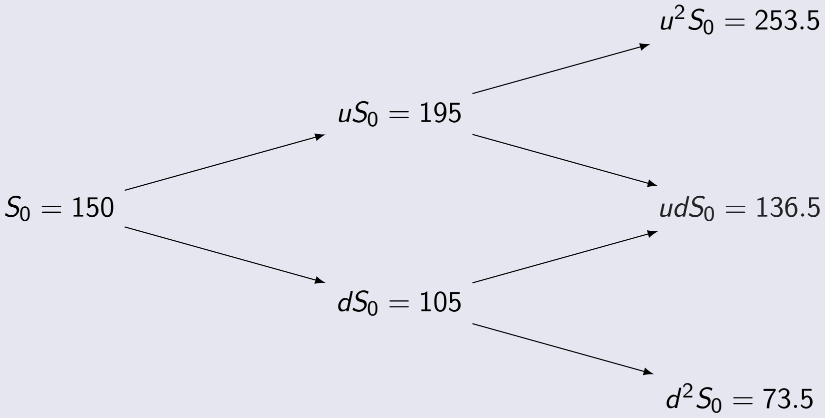
and

$$q_d = \frac{u - e^{rh}}{u - d} = 1 - q_u = 0.4916\,.$$

Binomial Model - Continuous Dividends

Example (European Put Option, Continuous Dividends)

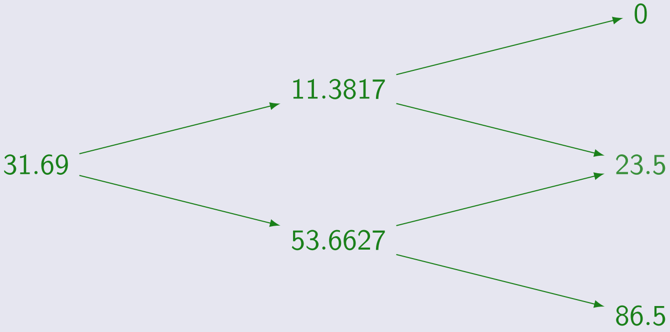
The two-period tree showing the evolution of the stock price is the same as before:



Binomial Model - Continuous Dividends

Example (European Put Option, Continuous Dividends)

As an exercise, verify that the price of the option is the following:



Binomial Model - Discrete Dividends

- For discrete dividends, we have incorporate the drops in stock prices into the tree itself.
- This will become clear in the following example.

Binomial Model - Discrete Dividends

Example (European Call Option, Discrete Dividend)

Suppose we are given the following information on an at-the-money European call option:

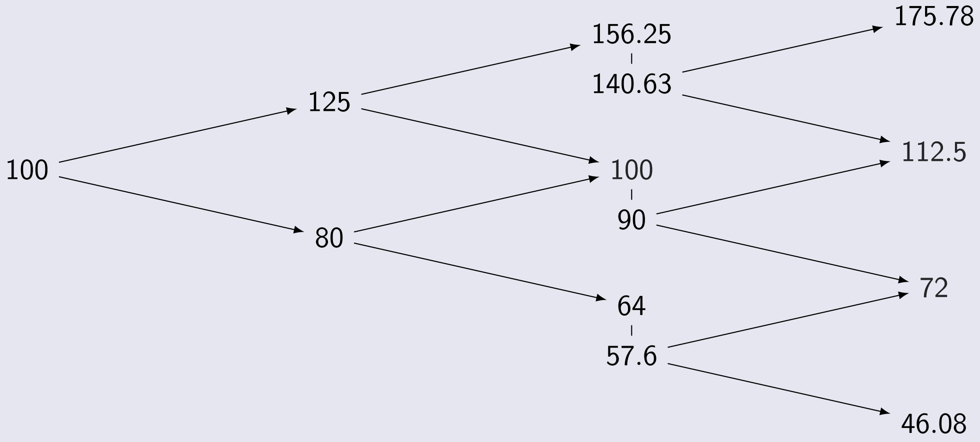
- The strike is $K = 100$ and the spot price is $S_0 = 100$.
- The time to expiry is $T = 3$ years.
- We use a 3-period model. The stock can either increase by 25% or decrease by 20% after one period.
- The risk free rate is $r = 0\%$.
- The stock pays a dividend of 10% of its cum-dividend price after 2 years.

What should the price of the option be?

Binomial Model - Discrete Dividends

Example (European Call Option, Discrete Dividend)

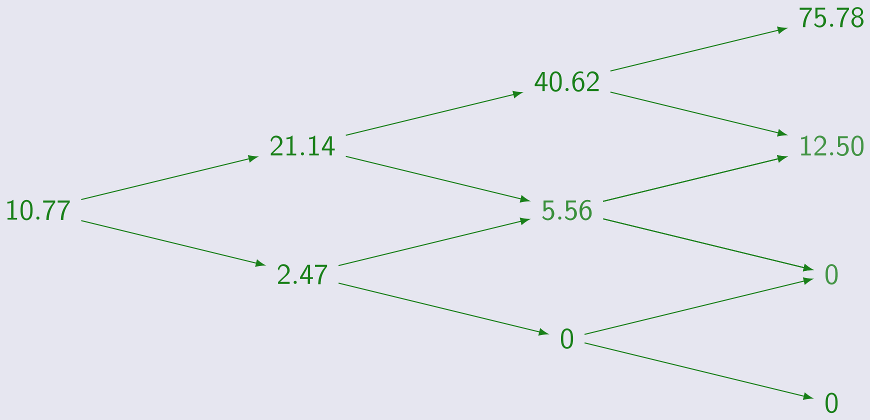
The stock price is given by the following:



Binomial Model - Discrete Dividends

Example (European Call Option, Discrete Dividend)

As an exercise, verify that $q_u = 4/9$, and that the price of the option is the following:



Binomial Model - Discrete Dividends

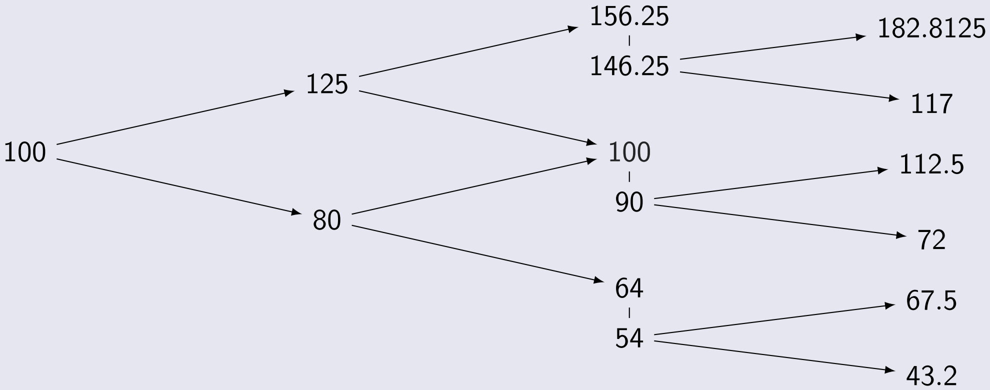
- So far, all our trees have "recombined".
- However, if the discrete dividend is not proportional to the stock price, then our tree will not recombine. We will see more examples of non-recombining trees in the following section.
- Nonetheless, option pricing is still possible using the same methods as before.

⇒ Suppose in the previous example, instead of having a discrete dividend of 10% of the stock price, the dividend was instead a fixed \$10.

Binomial Model - Discrete Dividends

Example (European Call Option, Discrete Dividend)

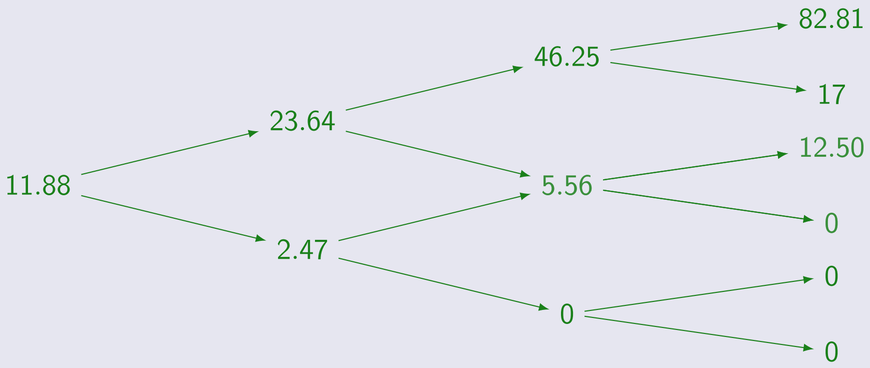
The stock price would be given by the following:



Binomial Model - Discrete Dividends

Example (European Call Option, Discrete Dividend)

As an exercise, verify that $q_u = 4/9$, and that the price of the option is the following:



Option Pricing in the Binomial Model: Exotic Options

Binomial Model - Exotic Options

- There are derivatives with more complicated payoffs than those of calls and puts.
- A lot of these derivatives are path-dependent, which means that the payoffs depend on the path of the stock price over a period of time.
  - In other words, $X = \Phi(S_0, S_h, \ldots, S_T)$. This does not always simplify to $X = \Phi(S_T)$, as it does in the case of calls and puts with continuous dividends.
  - We saw that we need to consider different paths when considering discrete dividends. It is important to recognize when all paths need to be considered.
- Path-dependent options include Asian options, barrier options, lookback options, etc.
- Exotic options are created for complex risk/portfolio management purposes, and are generally traded over-the-counter.

Binomial Model - Asian Options

Definition (Asian Options)

An Asian option has a payoff that depends on the average of the stock price over the time period. For example, with $k = 3$ and $h = 1$ year:

- An Asian call option with strike K has payoff

$$\max\left\{\frac{S_1 + S_2 + S_3}{3} - K, 0\right\}.$$

- An Asian put option with strike K has payoff

$$\max\left\{K - \frac{S_1 + S_2 + S_3}{3}, 0\right\}.$$

$\Rightarrow$ The Asian option can still be priced using recursive evaluation. However, since it is path-dependent, all 2^k paths of the tree must be drawn.

Binomial Model - Asian Option

Example (Asian Option)

Suppose we are given the following information on a binomial model with $k = 3$ time periods:

- The spot price is $\$200$.
- The time to expiry is $T = 3$ years.
- We have $u = 1.2$ and $d = 0.9$.
- The risk-free rate is $r = 5\%$.

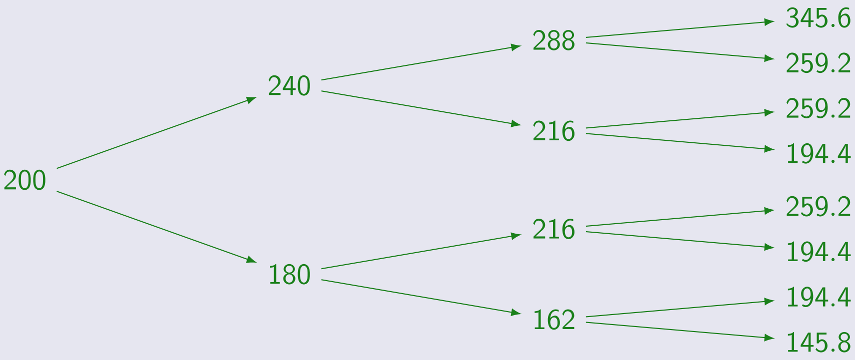
Determine the price of an Asian put option with strike $K = \$200$. Its payoff is

$$\max\left\{200 - \frac{S_1 + S_2 + S_3}{3}, 0\right\}.$$

Binomial Model - Asian Option

Example (Asian Option)

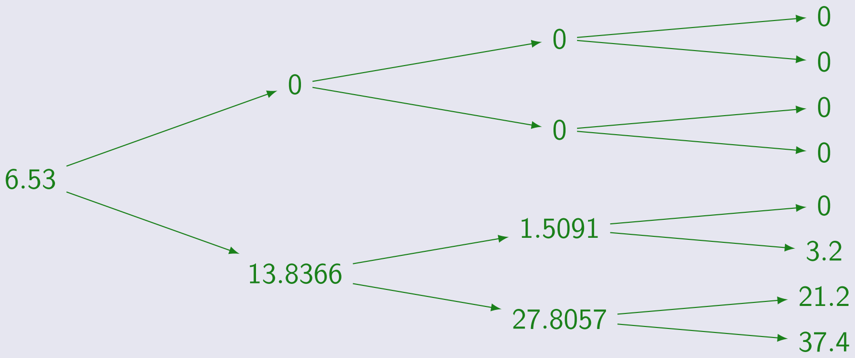
For path-dependent options, draw all possible paths of the stock price:



Binomial Model - Asian Option

Example (Asian Option)

As an exercise, verify that $q_u = 0.5042$, and that the price of the option is the following:



Binomial Model - Barrier Option

Definition (Barrier Options)

An barrier option has a payoff that depends on whether or not the stock price reaches a given level over the time period.

- A knock-out option pays a fixed rebate or becomes worthless if the stock price hits a given barrier before expiration. Otherwise, its payoff is the same as that of a standard option.
- A knock-in option pays a fixed rebate or becomes worthless if the stock price does not hits a given barrier before expiration. Otherwise, its payoff is the same as that of a standard option.

$\Rightarrow$ For example, a knock-out call option with no rebate pays 0 if the barrier is reached, and $\max\{S_T - K, 0\}$ otherwise.

Binomial Model - Barrier Option

Example (Barrier Option)

Suppose we are given the following information on a binomial model with $k = 3$ time periods:

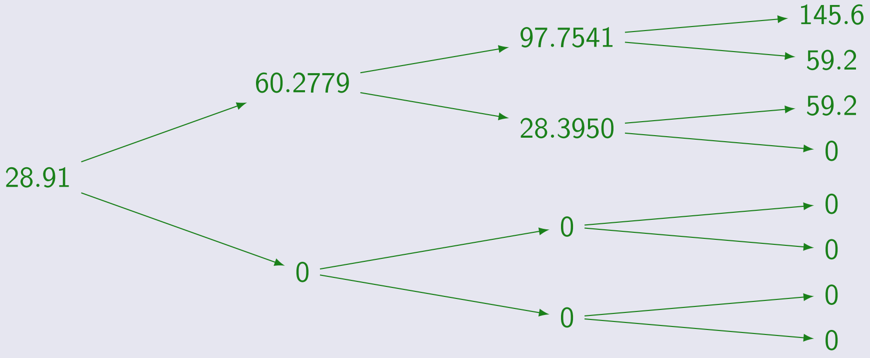
- The spot price is \$200.
- The time to expiry is $T = 3$ years.
- We have $u = 1.2$ and $d = 0.9$.
- The risk-free rate is $r = 5\%$.

Determine the price of a no-rebate knock-out call option with strike $K = \$200$ and barrier \$190.

Binomial Model - Barrier Option

Example (Barrier Option)

As an exercise, verify that the price of the option is the following:



Binomial Model - Lookback Option

Definition (Lookback Options)

A lookback option has a payoff that depends on the maximum or minimum price of the stock during the life of an option.

- A lookback call (floating strike) pays $\max\{S_T - S_{\min}, 0\}$ at expiration time T.
- A lookback put (floating strike) pays $\max\{S_{\max} - S_T, 0\}$ at expiration time T.
- A lookback call (fixed strike) pays $\max\{S_{\max} - K, 0\}$ at expiration time T.
- A lookback put (fixed strike) pays $\max\{K - S_{\min}, 0\}$ at expiration time T.

$\Rightarrow$ Here, $S_{\min}$ and $S_{\max}$ denote the minimum and maximum prices the stock achieved before expiration.

Binomial Model - Lookback Option

Example (Lookback Option)

Suppose we are given the following information on a binomial model with $k = 3$ time periods:

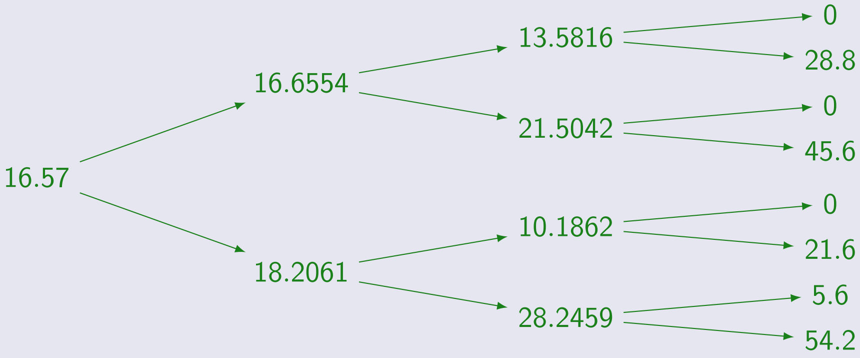
- The spot price is \$200.
- The time to expiry is $T = 3$ years.
- We have $u = 1.2$ and $d = 0.9$.
- The risk-free rate is $r = 5\%$.

Determine the price of a floating-strike lookback put option.

Binomial Model - Lookback Option

Example (Lookback Option)

As an exercise, verify that the price of the option is the following:



General Discrete-Time Models

General Model - Introduction

- So far, we have examined the binomial model. Recall that for the one-period binomial model:
  - There were only two states: "up" and "down".
  - There were only two assets: the stock and the risk-free bond.
- We now seek to generalize beyond these assumptions. We start by examining a single period market model (time 0 to h) such that:
  - There can be more than two states of the market.
  - There can be more than one risky asset.
- As before, we will try to price securities by replicating payoffs and applying the no-arbitrage principle.
- Also as before, risk-neutral valuation will be possible under certain conditions.

General Model - Introduction

- Under a more formal probability framework, a probability space is defined in terms of a state space
$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_M\},$$
with each ω_j representing a possible future state of the market, for $j = 1, \ldots, M$.

- We assume there is a probability measure $\mathbb{P}$ on the set Ω. That is $\mathbb{P}(\omega_j)$ is the probability that the future state ω_j will occur. We assume that $\mathbb{P}(\omega_j) > 0$.

- A random variable is a (measurable) function from Ω to $\mathbb{R}$.

- It is straightforward to verify that in the one-period binomial model, we have $M = 2$ states, and that S_h is a random variable.

Single-Period Market Model - Setup

- Now, let us consider N securities with random future prices. We will use the notation $S_t^{(i)}$ to represent the price of the i-th security at time t.
- As before, $S_0^{(i)}$ denotes the current price of security i, which is known. The price of security at the end of one period is $S_h^{(i)}$, which is a random variable.
- We can write all possible values of $S_h^{(i)}$ in the form of a vector. For each $i = 1, \ldots, N$, define

$$S_h^{(i)} := \begin{bmatrix} S_h^{(i)}(\omega_1) \\ S_h^{(i)}(\omega_2) \\ \vdots \\ S_h^{(i)}(\omega_M) \end{bmatrix}.$$

Single-Period Market Model - Setup

- Since there are N total securities, we can also define the time-t prices of all securities as a vector:

$$S_t := \begin{bmatrix} S_t^{(1)} & S_t^{(2)} & \dots & S_t^{(N)} \end{bmatrix}.$$

- Combining this with the vector representation of $S_h^{(i)}$ on the previous slide, all possible prices after one period are represented by the following $M \times N$ matrix:

$$S_h(\Omega) := \begin{bmatrix} S_h^{(1)}(\omega_1) & S_h^{(2)}(\omega_1) & \dots & S_h^{(N)}(\omega_1) \\ S_h^{(1)}(\omega_2) & S_h^{(2)}(\omega_2) & \dots & S_h^{(N)}(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ S_h^{(1)}(\omega_M) & S_h^{(2)}(\omega_M) & \dots & S_h^{(N)}(\omega_M) \end{bmatrix}.$$

- Here, each row represents a different future state of the world. Each column represents a different security.

Single-Period Market Model - Setup

- It will often be useful for the first security $S^{(1)}$ to represent a risk-free asset. If so, then:
  - $S_0^{(1)} = 1$, and,
  - $S_h^{(1)}(\omega) = e^{rh}$ for each $\omega \in \Omega$.
- For the one-period binomial model,

$$S_0 := \begin{bmatrix} 1 & S_0^{(2)} \end{bmatrix} \qquad S_h(\Omega) := \begin{bmatrix} e^{rh} & S_0^{(2)} u \\ e^{rh} & S_0^{(2)} d \end{bmatrix}.$$

Single-Period Market Model - Setup

- Similar to before, investors can form portfolios of the N securities in fixed proportions.

Definition (Portfolio, Single-Period General Model)

A portfolio is a column vector of real numbers $\theta = \begin{bmatrix}\theta_1 & \theta_2 & \dots & \theta_N\end{bmatrix}^T \in \mathbb{R}^N$, which represents a position of θ_i units of asset i.

- Let $\{V_t^\theta\}_{t \geq 0}$ denote the value process of a portfolio. Then a portfolio θ has value process

$$V_t^\theta = \sum_{i=1}^{n} \theta_i S_t^{(i)} = S_t \cdot \theta \, .$$

Single-Period Market Model - Example

Example

Suppose $M = N = 3$, $S_0 = \begin{bmatrix}1 & 20 & 50\end{bmatrix}$, and

$$S_h(\Omega) = \begin{bmatrix}1.1 & 25 & 100 \\ 1.1 & 15 & 20 \\ 1.1 & 30 & 30\end{bmatrix}.$$

Let $\theta = \begin{bmatrix}200 & -10 & 3\end{bmatrix}^T$. Calculate V_0^θ and $S_h(\omega_j) \cdot \theta$ for each $j = 1, 2, 3$, and give an interpretation for each of these values. Here, $S_h(\omega_j)$ refers to the j-th row of $S_h(\Omega)$.

Single-Period Market Model - Arbitrage

- We are now ready to define an arbitrage opportunity in this context.

Definition (Arbitrage Opportunity)

An arbitrage opportunity is a portfolio such that:

(i) $S_0 \cdot \theta \leq 0$, and,

(ii) $S_h(\Omega) \cdot \theta > 0$.

- Here, $S_h(\Omega) \cdot \theta > 0$ means that all its components are ≥ 0, and at least one component is > 0.

- It can be verified that this is exactly the same definition as in the one-period binomial model. That is, there is arbitrage if we can make a strictly positive profit with positive probability.

Single-Period Market Model - Arbitrage

- As always...

Definition (Principle of No Arbitrage)

There are no arbitrage opportunities in this market.

- For the one-period binomial model, we proceeded to prove that the Principle of No Arbitrage holds if and only if there exists a risk-neutral measure (FTAP 1).
- However, in this general market model, there might not be a risk-free asset! Therefore we cannot prove the same result for this model. In the following, we will try to prove a similar result that will hopefully also be useful for pricing.

Single-Period Market Model - State-price Vector

Definition (State-price Vector)

A state-price vector Ψ is a strictly positive row vector

$$\Psi = \begin{bmatrix} \Psi_1 & \Psi_2 & \dots & \Psi_M \end{bmatrix}$$

such that $S_0 = \Psi \cdot S_h(\Omega)$. Equivalently, for all $i = 1, \dots, N$, we have

$$S_0^{(i)} = \sum_{j=1}^{M} \Psi_j S_h^{(i)}(\omega_j) .$$

- If we want to find a state-price vector, then we would need to solve a system of N equations and M unknowns.
- If $N > M$, then there may not exist a state-price vector.
- If $M > N$, then there might be more than one state-price vector. We see that state-price vectors is related to the rank of the matrix $S_h(\Omega)$.

Single-Period Market Model - Example

Example

Consider a single-period market model with $S_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and

$$S_h(\Omega) = \begin{bmatrix} 2 & 0 \\ 2 & 4 \\ 2 & 3 \end{bmatrix}.$$

Show that the collection of all state-price vectors is given by

$$\Psi = \begin{bmatrix} \frac{1}{4} - \frac{1}{4}x, & \frac{1}{4} - \frac{3}{4}x, & x \end{bmatrix}$$

where $x \in (0, 1/3)$.

Single-Period Market Model - FTAP

Theorem (First Fundamental Theorem of Asset Pricing (FTAP 1))

In the single-period market model, The Principle of No Arbitrage holds if and only if there exists a state-price vector.

Proof.

We will only show that if there exists a state-price vector, then there is no arbitrage. Let θ be a portfolio, and suppose that $S_h(\Omega) \cdot \theta > 0$. Then since Ψ is strictly positive, we have

$$S_0 \cdot \theta = \Psi \cdot S_h(\Omega) \cdot \theta > 0\,.$$

Hence, arbitrage is not possible. The other direction of the proof requires the Hyperplane Separation Theorem and is beyond the scope of this course. □

Single-Period Market Model - Example

Example (Binomial Model)

Consider a single-period market model with $S_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and

$$S_h(\Omega) = \begin{bmatrix} e^{rh} & u \\ e^{rh} & d \end{bmatrix} .$$

Show that the only possible state-price vector is

$$\Psi = \begin{bmatrix} e^{-rh} q_u & e^{-rh} q_d \end{bmatrix} ,$$

and conclude that there exists a state-price vector if and only if $d < e^{rh} < u$.

Single-Period Market Model - Contingent Claims

- This time, a contingent claim with expiry date h is a random variable. We will use the letter X to denote the payoff of contingent claims, as before.
  - Note that before, our contingent claims were functions of terminal stock price: $X = \Phi(S_h)$. Since S_h is a random variable, so is X, so these two definitions do not contradict one another.
  - For each ω_j, the value $X(\omega_j)$ represents the payoff of X in state j. We can write X as a vector:

$$X = \begin{bmatrix} X(\omega_1) \\ X(\omega_2) \\ \vdots \\ X(\omega_M) \end{bmatrix}$$

- Our goal is to find a fair time-0 price for X.

Single-Period Market Model - Valuation

- The following result indicates what an arbitrage-free price should be.

Proposition

Suppose we have an arbitrage-free single-period market model, defined by S_0 and $S_h(\Omega)$. Let X be a contingent claim. Then X_0 is an arbitrage-free price in this market if and only if

$$X_0 = \Psi \cdot X \, ,$$

where Ψ is a state-price vector.

- Note that in this case, the arbitrage-free price may not be unique!
- Recall that if the binomial model is arbitrage-free, then it is complete. This is not true in general, so we need to consider these two concepts separately.

Single-Period Market Model - Valuation

Proof.

Consider the market with $N+1$ assets that includes the contingent claim X. Then by FTAP, this market is arbitrage-free if and only if there exists a state-price vector. That is, there exists Ψ such that for each $i = 1, \ldots, N+1$,

$$S_0^{(i)} = \sum_{j=1}^{M} \Psi_j S_h^{(i)}(\omega_j)\,.$$

Note that the first N equations are exactly the conditions for Ψ to be a state-price vector in the original market. The $(N+1)$-st equation is exactly

$$X_0 = \Psi \cdot X\,,$$

which concludes the proof. □

Single-Period Market Model - Example

Example

Consider a single-period market model with $S_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and

$$S_h(\Omega) = \begin{bmatrix} 2 & 0 \\ 2 & 4 \\ 2 & 3 \end{bmatrix}.$$

Let X be a call option on $S^{(2)}$ with strike 2 and expiration h. Show that the possible arbitrage-free prices are given by the interval $(1/3, 1/2)$.

Next, assume that the price of this call option is $2/5$. Find all state-price vectors in the market (with this call option included).

Single-Period Market Model - Completeness

Definition (Attainability)

A contingent claim X is attainable if there exists a portfolio θ such that

$$V_h^\theta = S_h \cdot \theta = X \,.$$

In this case, θ is a replicating portfolio for X.

Definition (Completeness)

A market is complete if it is arbitrage-free and all contingent claims are attainable.

- A convenient characterization of completeness in this market is given on the following slide.

Single-Period Market Model - Completeness

Lemma

An arbitrage-free single-period market model is complete if and only if the $M \times N$ matrix $S_h(\Omega)$ has rank M.

Proof.

Let X be a contingent claim. Then θ is a replicating portfolio if it satisfies the equation

$$S_h(\Omega) \cdot \theta = X \,.$$

This equation has a solution for arbitrary X if and only if the image of $S_h(\Omega)$ is $\mathbb{R}^M$. That is, $S_h(\Omega)$ has rank M. □

- As a corollary, we see that the binomial model is complete, since its matrix (as given in an earlier example) has rank 2.

Single-Period Market Model - Completeness

Theorem (Second Fundamental Theorem of Asset Pricing (FTAP 2))

An arbitrage-free single-period market model is complete if and only if the state-price vector is unique.

Proof.

Recall that Ψ is a state-price vector if and only if it solves the equation

$$S_0 = \Psi \cdot S_h(\Omega)\,.$$

Taking transposes, this is equivalent to

$$S_0^T = S_h(\Omega)^T \cdot \Psi^T\,.$$

Then this equation has a unique solution if and only if the kernel of $S_h(\Omega)^T$ is $\{0\}$. $\square$

Single-Period Market Model - Completeness

Proof (Cont'd).

By the rank-nullity theorem, we have

$$\begin{aligned}\ker(S_h(\Omega)^T) = \{0\} &\iff \mathrm{Nullity}(S_h(\Omega)^T) = 0 \\ &\iff \mathrm{Rank}(S_h(\Omega)^T) = M \\ &\iff \mathrm{Rank}(S_h(\Omega)) = M \\ &\iff \text{The model is complete}\,.\end{aligned}$$

□

Single-Period Market Model - Risk-Neutral Measure

- It seems that the state-price vector is connected to the idea of a risk-neutral measure. Indeed, suppose that $S^{(1)}$ earns continuous interest at a rate r. That is, $S_0^{(1)} = 1$ and $S_h^{(1)}(\omega_j) = e^{rh}$ for all $j = 1, \ldots, M$.
- Let Ψ be a state-price vector.
- Define $\mathbb{Q}(\omega_j) = e^{rh}\Psi(\omega_j)$. Then we can verify that $\mathbb{Q}$ is indeed a measure on Ω:

$$1 = S_0^{(1)} = \Psi \cdot S_h^{(1)} = \sum_{j=1}^{M} \Psi(\omega_j) e^{rh} = \sum_{j=1}^{M} \mathbb{Q}(\omega_j) e^{-rh} e^{rh} = \sum_{j=1}^{M} \mathbb{Q}(\omega_j)\,.$$

- This $\mathbb{Q}$ is a risk-neutral measure. Note that if X is a contingent claim, then its time-0 price satisfies

$$X_0 = \Psi \cdot X = e^{-rh} \mathbb{E}^{\mathbb{Q}}[X]\,.$$

Single-Period Market Model - Numeraire

- In general, the risk-neutral measure is what we are usually interested in. More generally, if there is no risk-free asset in the market, we can still find a "risk-neutral measure" by designating the first asset $S^{(1)}$ as the numeraire.
- For each $i = 1, \dots, N$ and each time t, let $Z_t^{(i)}(\omega_j) = \frac{S_t^{(i)}(\omega_j)}{S_t^{(1)}(\omega_j)}$. Then $Z_0^{(1)} = 1$, and we see that the whole first column of $Z_h(\Omega)$ is 1.
- Essentially, we are expressing everything in terms of the numeraire $S^{(1)}$.
- All the theory still applies to the market determined by Z_0 and $Z_h(\Omega)$, but now a state-price vector Ψ is actually a measure, by defining $\mathbb{Q}(\omega_j) = \Psi(\omega_j)$.
- This $\mathbb{Q}$ is also known as the martingale measure. If X is expressed in terms of the numeraire, then

$$X_0 = \mathbb{E}^{\mathbb{Q}}[X_h].$$

Single-Period Market Model - Example

Example (Binomial Model)

Consider a single-period market model with $S_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and

$$S_h(\Omega) = \begin{bmatrix} e^{rh} & u \\ e^{rh} & d \end{bmatrix}.$$

Take the first asset as the numeraire. Show that the unique martingale measure is given by $\mathbb{Q}(\omega_1) = q_u$ and $\mathbb{Q}(\omega_2) = q_d$.

- Note that if we solve directly for Ψ and calculate the risk-neutral measure as $\mathbb{Q} = e^{rh}\Psi$, we would get the same thing. However, using a numeraire to find a martingale measure is what is usually done in practice.

Multiperiod Market Model

- Although beyond the scope of this course, the general market model can be extended to multiple periods.
- Similar to the multiperiod binomial model, we will need to consider self-financing portfolio strategies, and modify the definition of arbitrage.
- In the multiperiod setting, FTAP 1 and FTAP 2 will still hold.
- Risk-neutral pricing will be possible as well, using a martingale measure defined with respect to a choice of numeraire. If X is expressed in terms of the numeraire, then the following formula still applies:

$$X_0 = \mathbb{E}^{\mathbb{Q}}\left[X_T\right].$$